Queen Mary
University of London

Main Examination period 2017

MTH716U / MTHM007: Measure Theory and Probability

Duration: 3 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Examiners: C. H. Joyner

Throughout this exam the term **measurable** will be used to mean **Lebesgue measurable** and $\mathscr{M}$ will denote the collection of Lebesgue measurable subsets of $\mathbb{R}$. For all measurable sets $E \in \mathscr{M}$ we will denote $m(E)$ to be the corresponding Lebesgue measure of E.

Question 1. [25 marks]

(a) State the definition of a **null set** $A \subset \mathbb{R}$. [**3**]

(b) The **outer measure** of a set $A \subseteq \mathbb{R}$ is denoted $m^*(A)$ and defined by

$$m^*(A) := \inf Z_A , \qquad Z_A := \left\{ \sum_{n=1}^{\infty} l(I_n) : A \subseteq \bigcup_{n=1}^{\infty} I_n,\ I_n \text{ are intervals} \right\} .$$

Show that we obtain an equivalent definition if ‘intervals’ is replaced by ‘open intervals’ in the above definition. [**6**]

(c) Show that for two sets $A \subseteq B \subseteq \mathbb{R}$ we have the **monotonicity** condition $m^*(A) \leq m^*(B)$. [**4**]

(d) Show that outer measure is **countably sub-additive**, i.e. for sets $A_1, A_2, \ldots \subseteq \mathbb{R}$ the relation

$$m^* \left(\bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} m^*(A_n)$$

is satisfied. [**7**]

(e) Prove that for any constant $c \geq 0$ and set $A \subseteq \mathbb{R}$ the outer measure obeys

$$m^*(cA) = cm^*(A),$$

where $cA := \{cx : x \in A\}$. [**5**]

Question 2. [25 marks]

(a) State the definition of a **measurable set** $E \subseteq \mathbb{R}$. [**3**]

(b) Using this definition, show that any **null set** is measurable. You may use that the outer measure of a null set $A \subset \mathbb{R}$ satisfies $m^*(A) = 0$. [**3**]

(c) Show that for any measurable set $E \in \mathscr{M}$ and constant $c \geq 0$ the set cE is also measurable. You may use that outer measure satisfies $m^*(cA) = cm^*(A)$ for all $c \geq 0$ and $A \subseteq \mathbb{R}$. [**4**]

(d) State what it means for Lebesgue measure to be **countably additive** and then use this to show that if $E_1, E_2, \ldots \in \mathscr{M}$ are a sequence of measurable sets such that $E_n \subset E_{n+1}$ for all n, then

$$m\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m(E_n).$$

[6]

(e) State the three properties required for $\mathscr{F}$ to be a σ-field on the set $\Omega \subseteq \mathbb{R}$. **[2]**

(f) Suppose we have a game in which the outcome is either a win (W), a loss (L) or a draw (D).

(i) Let $\mathscr{F}$ be the collection of all subsets of $\Omega = \{W, L, D\}$. Show that $\mathscr{F}$ is a σ-field. **[3]**

(ii) What is the σ-field $\mathscr{F}_W$ generated by the event $\{W\}$? **[2]**

(iii) Give an example of a probability measure on the σ-fields $\mathscr{F}$ and $\mathscr{F}_W$. **[2]**

Question 3. [20 marks]

(a) Give the definition of a **measurable function** $f : \mathbb{R} \to \mathbb{R}$. **[3]**

(b) State an **alternative equivalent definition** to the one provided in Part (a). **[3]**

(c) Suppose that f is a measurable function. Using your answer to Part (b), or otherwise, show that the truncation of f, given by

$$f^a(x) = \begin{cases} a & \text{if } f(x) > a \\ f(x) & \text{if } f(x) \le a, \end{cases}$$

is also measurable. **[3]**

(d) A function f is said to be monotonically increasing if for all $x < y$ it satisfies $f(x) \le f(y)$. Using your answer to Part (b), or otherwise, show that every monotonically increasing function is measurable. **[5]**

(e) State the definition of a **random variable** X on a probability space $(\Omega, \mathscr{F}, P)$. **[2]**

(f) Let $X : [0,1] \to \mathbb{R}$ be the random variable given by

$$X(\omega) = \frac{1}{2}\mathbf{1}_{[0,\frac{1}{3})}(\omega) + \frac{3}{2}\mathbf{1}_{[\frac{1}{3},\frac{2}{3}]}(\omega) + \mathbf{1}_{(\frac{2}{3},1]}(\omega),$$

where $\mathbf{1}_E(\omega)$ denotes the indicator function over a set $E \subseteq \mathbb{R}$. What is the σ-field generated by X? **[4]**

Question 4. [30 marks]

(a) State the definition of a **simple function** ϕ and its **Lebesgue integral** $\int_E \phi \, dm$ for a measurable set $E \subseteq \mathbb{R}$. **[2]**

(b) State the definition of the **Lebesgue integral** $\int_E f \, dm$ for a **non-negative measurable function** f and measurable set $E \subseteq \mathbb{R}$. **[2]**

(c) Let f and g be two non-negative measurable functions and suppose $f \leq g$ on the measurable set $E \subseteq \mathbb{R}$. Show that

$$\int_E f \, dm \leq \int_E g \, dm.$$

[4]

(d) Let f be a non-negative measurable function and $E \subseteq \mathbb{R}$ a measurable set such that $a \leq f(x) \leq b$ for all $x \in E$. Using Part (c), or otherwise, show that

$$am(E) \leq \int_E f \, dm \leq bm(E).$$

[3]

(e) (i) State what it means for a function f to be equal to zero **almost everywhere** on $\mathbb{R}$. **[2]**

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative measurable function. Using Part (c), or otherwise, show that if $\int_{\mathbb{R}} f \, dm = 0$ then $f = 0$ almost everywhere on $\mathbb{R}$. **[4]**

(f) State Fatou's Lemma for a sequence of measurable functions $\{f_n\}$. **[3]**

(g) State the Monotone Convergence Theorem. **[4]**

(h) Use Fatou's Lemma to prove the Monotone Convergence Theorem. **[6]**

End of Paper.