MTH716U / MTHM007: Measure Theory and Probability

Duration: 3 hours

Date and time: 6th May 2016, 14:30-17:30

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

Possession of unauthorised material at any time when under examination conditions is an assessment offence and can lead to expulsion from QMUL. Check now to ensure you do not have any notes, mobile phones, smartwatches or unauthorised electronic devices on your person. If you do, raise your hand and give them to an invigilator immediately. It is also an offence to have any writing of any kind on your person, including on your body. If you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms it shall be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. A mobile phone that causes a disruption in the exam is also an assessment offence.

Exam papers must not be removed from the examination room.

Examiner(s): C. H. Joyner

Throughout this exam the term **measurable** will be used to mean **Lebesgue measurable**

Question 1.

(a) State the definition of a null set. **[2]**

(b) Show that a countable union of null sets is also null. **[6]**

(c) Given a subset $A \subseteq \mathbb{R}$, how is the **outer measure** $m^*(A)$ defined? **[3]**

(d) Prove that A is null if and only if $m^*(A) = 0$. **[5]**

(e) Show that for two sets $A \subseteq B \subseteq \mathbb{R}$ we have the **monotonicity** condition $m^*(A) \leq m^*(B)$. **[4]**

(f) What does it mean to say that outer measure is **sub-additive**? **[2]**

(g) Using Parts (d) and (e) and assuming outer measure is sub-additive show that if A is null then $m^*(A \cup B) = m^*(B)$ for any B. **[3]**

Question 2.

(a) State the definition of a measurable set $E \subseteq \mathbb{R}$. **[3]**

(b) Explain (but do not prove) why $\mathcal{M}$, the collection of all measurable sets on $\mathbb{R}$, is a σ-field. **[3]**

(c) Using Part (b) show that $\mathcal{M}$ is closed under countable intersections, i.e. for $E_1, E_2, \ldots \in \mathcal{M}$ the set $\bigcap_{n=1}^{\infty} E_n \in \mathcal{M}$. **[3]**

(d) Explain why $\mathcal{M}$ restricted to $[0, 1]$, the collection of measurable subsets of $[0, 1]$, is also a σ-field. You may assume that $[0, 1]$ is measurable. **[2]**

(e) Show that if $A, B \in \mathcal{M}$ and $A \subset B$ with $m(A) < \infty$ then $m(B \setminus A) = m(B) - m(A)$. **[2]**

(f) Using the definition of outer measure show that for all $A \subset \mathbb{R}$ and $\epsilon > 0$ there exists an open set $O \supset A$ such that $m(O) \leq m^*(A) + \epsilon$. Hence, using Part (e), show that for all $E \in \mathcal{M}$ there is an open set $O \supset E$ such that $m(O \setminus E) < \epsilon$. **[7]**

(g) Using Part (f) show that for any $E \in \mathcal{M}$ one may find a sequence of open sets $\{O_n\}$ such that

$$E \subset O = \bigcap_{n=1}^{\infty} O_n, \qquad m\left(\bigcap_{n=1}^{\infty} O_n\right) = m(E).$$

[5]

Question 3.

(a) Give the definition of a measurable function $f : \mathbb{R} \to \mathbb{R}$. **[3]**

(b) State an alternative equivalent definition to the one provided in Part (a). **[3]**

(c) What does it mean to say a function f is equal to 0 almost everywhere on a set E? **[2]**

(d) Using your answers to Parts (b) and (c), or otherwise, show that a function $f : E \to \mathbb{R}$ which is equal to 0 almost everywhere is measurable. **[3]**

(e) If $f_n : \mathbb{R} \to \mathbb{R}$ is a sequence of measurable functions, show that the following are measurable

(i) $\max_{n \leq k} f_n$

(ii) $\min_{n \leq k} f_n$ **[4]**

Question 4.

(a) Explain why the function ϕ, given by

$$\phi(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$$

is simple and evaluate its Lebesgue integral $\int_{\mathbb{R}} \phi \, dm$. **[4]**

(b) Given a measurable set E state the definition of the integral $\int_E f \, dm$ for a non-negative measurable function $f : E \to \mathbb{R}$. **[3]**

(c) What does it mean to say that a measurable function f is integrable over a measurable set $E \subseteq \mathbb{R}$? **[3]**

(d) Give an example where f and g are integrable over $E = [0, 1]$ but the product fg is not. **[4]**

(e) State Fatou's Lemma for a sequence of measurable functions $\{f_n\}$. **[3]**

(f) State the Dominated Convergence Theorem. **[4]**

(g) Use Fatou's Lemma to prove the Dominated Convergence Theorem. **[10]**

(h) Use the Dominated Convergence Theorem to evaluate

$$\lim_{n \to \infty} \int_1^\infty \frac{x}{1 + nx^3} \, dx.$$

[4]

End of Paper.