Queen Mary
University of London

M. Sci. Examination by course unit 2014

MTH716U Measure Theory & Probability

Duration: 3 hours

Date and time: 28th April 2014, 10:00–13:00

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt all questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

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Examiner(s): I.Goldsheid

Standing Assumptions: $(\Omega, \mathscr{F}, m)$ **denotes a measure space;** $f : \Omega \to \mathbb{R}$ **and** $g : \Omega \to \mathbb{R}$ **are real valued functions on** Ω**.**

Question 1 In questions 1(a)–1(e), the term *measurable* means *Lebesgue measurable*.

(a) Given a subset $A \subseteq [a,b]$, how is its *outer measure* $m^*(A)$ defined? [3]

(b) Use the definition of m^* to prove that if $A_1, A_2, A_3, \dots$ are subsets of $[a,b]$, then [6]

$$m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} m^*(A_n).$$

(c) Give the definition of a *null set*. Prove that if B is a null set then $m^*(A \cup B) = m^*(A)$. [4]

(d) When do we say that a subset $A \subset [a,b]$ is measurable? [1]

(e) Prove that open and closed subsets of $[a,b]$ are measurable. [9]

You may assume without proof that
i) every open set in $\mathbb{R}$ is a countable union of disjoint open intervals,
ii) $m^*(\cup_{j=1}^{n} I_j) = \sum_{j=1}^{n} l(I_j)$, where $\{I_j,\, 1 \leq j \leq n\}$ is a collection of disjoint intervals.

(f) Let m be a σ-additive measure on $(\Omega, \mathscr{F})$. Assuming that $m(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} m(A_n)$ whenever $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$ are measurable, prove that if $B_1 \supseteq B_2 \supseteq B_3 \supseteq \dots$ are measurable and $m(B_1) < \infty$, then [5]

$$m\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} m(B_n).$$

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Question 2 (a) Give the definition of a σ-field of subsets of a set Ω. [2]

(b) Consider a measurable space $(\Omega, \mathscr{F})$ and a function $f : \Omega \to \mathbb{R}$. Prove that the following two definitions of measurability of f are equivalent:
i) f is measurable if $\{\omega : f(\omega) \in (a,b)\} \in \mathscr{F}$ for any $a, b \in \mathbb{R}$.
ii) f is measurable if $\{\omega : f(\omega) < a\} \in \mathscr{F}$ for all $a \in \mathbb{R}$. [5]

(c) Let $f : \Omega \to \mathbb{R}$ and $g : \Omega \to \mathbb{R}$ be measurable functions. Prove that $f + g$ is measurable. [5]

(d) Prove that if $f_n : \Omega \to \mathbb{R},\ n = 1, 2, \dots$ are measurable functions then $\liminf_{n\to\infty} f_n$ is a measurable function. [5]

(e) Let $f_n : \Omega \to \mathbb{R},\ n = 1, 2, \dots$, be a sequence of measurable functions. Prove that the set of points in Ω where $\lim_{n\to\infty} f_n$ exists is measurable. You may assume without proof that $\liminf_{n\to\infty} f_n$ and $\limsup_{n\to\infty} f_n$ are measurable functions. [5]

Question 3 (a) Define what is the integral of a non-negative measurable function $f : \Omega \to \mathbb{R}$. [3]

(b) State and prove the Monotone Convergence Theorem. [10]

(c) Use the Monotone Convergence Theorem to prove the Fatou Lemma. [5]

(d) State (do not prove) the Beppo–Levi Theorem. [3]

(e) Use the Beppo-Levi Theorem, and the fact that $\sum_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}$, to prove that [5]

$$\int_0^\infty \frac{e^{-x}x}{1-e^{-x}}\,\mathrm{d}x = \frac{\pi^2}{6}\,.$$

Question 4 (a) Let $\mathscr{L}^1$ be the space of integrable functions on $(\Omega, \mathscr{F}, m)$.

Briefly explain how the space L^1 is defined and what is the definition of the norm of a function $f \in L^1$?

What is the definition of the distance between two functions f and g in a normed space X?

What is a Cauchy sequence of elements f_n, $n \geq 1$, of a normed space X?

What does it mean to say that a normed space X is complete? [5]

(b) Use the Beppo–Levi Theorem to prove the completeness of the space L^1. [10]

(c) Given two measures μ and ν on $(\Omega, \mathscr{F})$, what does it mean to say that μ is absolutely continuous with respect to ν? State the Radon–Nikodym Theorem. [4]

(d) Give the definition of a signed measure on $(\Omega, \mathscr{F})$. Explain what is the Hahn decomposition of Ω for a given signed measure. [5]

End of Paper