Queen Mary
University of London

M. Sci. Examination by course unit 2015

MTH714U: Group Theory

Duration: 3 hours

Date and time: 20 May 2015, 14.30h–17.30h

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and **cross through any work that is not to be assessed**.

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Examiner(s): L. H. Soicher

Question 1.

(a) What is meant by an *action* of a group G on a set Ω? **[3]**

(b) Consider the action by conjugation of $S = S_4$ on its conjugacy class C consisting of the elements with cycle structure $[2, 2]$, and let $c = (1, 2)(3, 4)$. For the given action, determine the following [you need not justify your answers]:

(i) the orbit $\mathrm{Orb}_S(c)$; **[2]**

(ii) the stabiliser $\mathrm{Stab}_S(c)$; **[2]**

(iii) the kernel of the action. **[2]**

(c) State the Orbit-Stabiliser Theorem. **[4]**

(d) Let G be a finite group and suppose $|G| = p^a.m$, where p is a prime, a is a non-negative integer, and m is a positive integer not divisible by p. Apply the Orbit-Stabiliser Theorem to prove that G has a subgroup of order p^a. [You may assume, without proof, that $\binom{p^a m}{p^a}$ is not divisible by p.] **[12]**

Question 2. Let G be a group, let Ω be a set with $|\Omega| > 1$, and suppose that G acts on Ω.

(a) What is meant by a G-*congruence* on Ω? What does it mean to say that G acts *transitively* on Ω, and what does it mean to say that G acts *primitively* on Ω? **[6]**

(b) Now suppose that n is an integer greater than 1 and G is a subgroup of the symmetric group S_n, acting naturally as permutations of $\{1, \ldots, n\}$.

(i) Define a relation $\equiv$ on $\{1, \ldots, n\}$ by $i \equiv j$ if and only if $i = j$ or the transposition (i, j) is an element of G. Prove that $\equiv$ is a G-congruence. **[9]**

(ii) Prove that S_n is generated by its set of transpositions. **[5]**

(iii) Deduce that if the action of G on $\{1, \ldots, n\}$ is primitive and G contains a transposition then $G = S_n$. **[5]**

Question 3.

(a) What is meant by a *permutation* of $\{1, \ldots, n\}$, what is meant by an *even* permutation of $\{1, \ldots, n\}$, what is meant by the *alternating group* A_n, and what does it mean to say that a group G is *simple*? **[8]**

(b) Prove that the alternating group A_5 is simple. [You may state, without proof, the sizes of the conjugacy classes of the elements of A_5.] **[7]**

(c) Prove that if G is a simple group of order 60 then $G \cong A_5$. [You may assume that A_6 is simple and is the only subgroup of index 2 in S_6.] **[10]**

Question 4. Suppose n is an integer greater than 1, F is a field, and $V = F^n$.

(a) Define the groups $\mathrm{GL}(n, F)$, $\mathrm{SL}(n, F)$, and $\mathrm{PSL}(n, F)$. **[3]**

(b) Give, without proof, the orders of the above groups, in the case where F is a finite field with q elements. **[3]**

(c) Explain why $\mathrm{PSL}(2, 4) \cong A_5$. **[5]**

(d) Let a be a non-zero vector in V. Define what is meant by a *transvection* $T(a, f)$ on V, and what is meant by the *transvection group* $A(a)$. **[4]**

(e) Let a be a non-zero vector in V and let $g \in \mathrm{GL}(n, F)$. Prove that $g^{-1}A(a)g = A(ag)$. **[10]**

End of Paper.