Queen Mary
University of London

Main Examination period 2021 – May/June – Semester B
Online Alternative Assessments

MTH6142: Complex Networks

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment:

- **You may use books and notes.**
- **You may use calculators and computers, but you must show your working for any calculations you do.**
- **You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.**
- **You must not seek or obtain help from anyone else.**

All work should be **handwritten** and should **include your student number**.

You have **24 hours** to complete and submit this assessment. When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

You are expected to spend about 2 hours to complete the assessment, plus the time taken to scan and upload your work. Please try to upload your work well before the end of the submission window, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

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Question 1 [40 marks].
Consider the adjacency matrix $\mathbf{A}$ of a network of size $N = 5$ given by

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

a) Draw the network. Is the network directed or undirected? (*Explain your answer.*) **[7]**

b) How many weakly and how many strongly connected components are there in the network? Which are the nodes belonging to each one of these components? **[4]**

c) Is there any in-component in the network? If yes, which are the nodes belonging to them? **[3]**

d) Is there any out-component in the network? If yes, which are the nodes belonging to it? **[3]**

e) Determine the in-degree sequence $\{k_1^{in}, k_2^{in}, k_3^{in}, k_4^{in}, k_5^{in}\}$ and the out-degree sequence $\{k_1^{out}, k_2^{out}, k_3^{out}, k_4^{out}, k_5^{out}\}$. **[4]**

f) Determine the in-degree distribution $P^{in}(k)$ and the out-degree distribution $P^{out}(k)$. **[4]**

g) Calculate the $N \times N$ matrix $\mathbf{d}$ of elements $d_{ij} \in \mathbb{N}_0 \cup \{\infty\}$ indicating the shortest distance of node j from node i. **[5]**

h) Calculate the eigenvector centrality x_i of each node $i = 1, 2, \ldots, N$ of the network with adjacency matrix $\mathbf{A}$ defined above.
To this end start from the initial guess $\mathbf{x}^{(0)} = \frac{1}{N}\mathbf{1}$ where $\mathbf{1}$ is the N-dimensional column vector of elements $1_i = 1 \; \forall i = 1, 2 \ldots, N$. Consider the iteration

$$\mathbf{x}^{(n)} = \mathbf{A}\mathbf{x}^{(n-1)},$$

for $n \in \mathbb{N}$.
Finally, calculate the eigenvector centrality x_i of each node i of the network by finding the limit

$$x_i = \lim_{n \to \infty} \frac{x_i^{(n)}}{\sum_{j=1}^{N} x_j^{(n)}}.$$

[10]

Question 2 [35 marks]. Consider the following model for a growing simple network.
We adopt the following notation: N and L indicate respectively the total number of nodes and links of the network, A_{ir} indicates the generic element of the adjacency matrix $\mathbf{A}$ of the network, k_i indicates the degree of node i and $\langle k \rangle$ indicates the average degree of the network.
At time $t = 1$ the network is formed by a $n_0 = 6$ nodes $m_0 = 6$ links.
At every time step $t > 1$ the network evolves according to the following rules:

- A link (r, s) between a node r and a node s is chosen randomly with uniform probability

$$\pi_{(r,s)} = \frac{A_{r,s}}{L}$$

  and is removed from the network.

- A single new node joins the network and is connected to the rest of the network by m links with m fixed to a time-independent integer constant satisfying $2 < m \leq 6$. Each of these new links connects the new node to a generic node j chosen with probability

$$\Pi_j = \frac{k_j}{\langle k \rangle N}.$$

a) Evaluate $\tilde{\Pi}_i(t)$ indicating the expected increase in the number of links of node i at any given time t and show that it follows the preferential attachment rule. **[8]**

b) What is the total number of links in the network at time t? What is the total number of nodes? **[2]**

c) What is the average degree $\langle k \rangle$ of the network at time t? What is the average degree in the limit $t \to \infty$? **[6]**

d) Use the result at point a) to derive the time evolution $k_i = k_i(t)$ of the average degree k_i of a node i for $t \gg 1$ in the mean-field, continuous approximation. **[6]**

e) Show that the degree distribution derived in the mean-field approximation is power-law in the limit of large network sizes. **[6]**

f) Indicate with γ the power-law exponent of the degree distribution found in the mean-field approximation. Derive the dependence of γ on m. **[3]**

g) For which values of m is the network scale-free? **[4]**

End of Paper.