Queen Mary
University of London

MTH6142 / MTH6142P: Complex networks

Duration: 2 hours

Date and time: 6th May 2016 10:00-12:00

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You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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Examiner(s): G. Bianconi

Question 1 (40 marks).

Structural properties of a given network.
Consider the adjacency matrix $\mathbf{A}$ of a directed network of size $N = 3$ given by

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

a) Draw the network. **[4]**

b) Does the network contain a strongly connected component? If yes, which nodes are part of the strongly connected component? **[2]**

c) Does the network contain an in-component? If yes, which nodes are part of the in-component? **[2]**

d) Does the network contain an out-component? If yes, which nodes are part of the out-component? **[2]**

e) Calculate the eigenvector centrality $\mathbf{x}$ of the nodes of the network with adjacency matrix $\mathbf{A}$ satisfying

$$\begin{aligned} \lambda_1 \mathbf{x} &= \mathbf{A}\mathbf{x}, \\ 1 &= \sum_i x_i, \end{aligned} \tag{1}$$

where λ_1 is the Perron-Frobenius eigenvalue of the matrix $\mathbf{A}$. **[13]**

f) Calculate the Katz centrality $\mathbf{x}$ satisfying

$$\mathbf{x} = \beta(\mathbb{I} - \alpha\mathbf{A})^{-1}\mathbf{1} \tag{2}$$

where $\alpha \in (0, 1/\lambda_1)$, $\beta > 0$, $\mathbb{I}$ is the identity matrix and $\mathbf{1}$ is the column vector of elements $1_i = 1 \; \forall i \in \{1, 2, 3\}$. **[16]**

g) Is there a difference in the ranking provided by the eigenvector centrality (part e)) and the Katz centrality (part f))? **[1]**

Question 2 (25 marks).

Random networks in the $\mathbb{G}(N, p)$ ensemble
Consider the network ensemble $\mathbb{G}(N, p)$ formed by all networks of N nodes with each pair of nodes connected with probability p.

a) Show that the degree distribution $P(k)$ of a generic network in this ensemble is a binomial given by

$$P(k) = \binom{N-1}{k} p^k (1-p)^{N-1-k}. \tag{3}$$

[**5**]

b) Calculate the generating function

$$G(x) = \sum_{k=0}^{N-1} P(k) x^k \tag{4}$$

for the binomial degree distribution $P(k)$ given by equation (3) [**6**]

c) Using the properties of the generating function, evaluate the first moment $\langle k \rangle$ and the second moment $\langle k(k-1) \rangle$ of the degree distribution $P(k)$ given by equation (3). [**4**]

d) Using the results of part c) calculate the variance σ^2 and the standard deviation σ of the degree distribution $P(k)$ given by equation (3). [**5**]

e) Calculate the average degree $\langle k \rangle$ of a random network in the ensemble $\mathbb{G}(N, p)$ with $N = 10001$ nodes, and the linking probability is $p = 10^{-2}$. [**2**]

f) Does the network of part e) have a giant component ? *Justify your answer.* [**3**]

Question 3 (35 marks).

Growing network model with uniform attachment
Consider the following model for a growing network with uniform attachment. At time $t = 0$ the network is formed by a connected network of $n_0 > m$ nodes.

- At every time step a single new node joins the network, so that at time t there will be exactly $N(t) = n_0 + t$ nodes. Every new node has initially m links.

- Each new link is attached to an existing node of the network. The target node i is chosen with probability Π_i following a uniform attachment rule $\Pi_i = \frac{1}{N(t)}$.

a) What is the time evolution $k_i = k_i(t)$ of the average degree k_i of a node i for $t \gg 1$ in the mean-field, continuous approximation? [**9**]

b) What is the degree distribution of the network at large times in the mean-field approximation? [**9**]

c) What is the average degree $\langle k \rangle$ of the network? [**4**]

d) Let $N^t(k)$ be the average number of nodes with degree k at time t.
Write the master equation satisfied by $N^t(k)$. [**4**]

e) Solve the master equation, finding the exact results for the degree distribution $P(k)$ in the limit $N \to \infty$. [**9**]

End of Paper.