Main Examination period 2018

MTH6140: Linear Algebra II

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Examiners: M. Jerrum, M. Fayers

Question 1. [20 marks] In this question, V is a finite-dimensional vector space over a field $\mathbb{K}$.

(a) Define what it means for a list $(v_1, \ldots, v_n)$ of vectors in V to be (i) **linearly independent**, (ii) **spanning**, and (iii) a **basis**. **[6]**

(b) Which of the following statements are true in general and which false? (No explanation is required.)

(i) Every basis of V has the same cardinality.

(ii) V has a unique basis up to reordering of vectors.

(iii) If $(v_1, \ldots, v_n)$ is a basis and $w \in V$ is any vector, then $(v_1 + w, \ldots, v_n + w)$ is a basis.

(iv) If $(v_1, \ldots, v_n)$ is a basis and $c \in \mathbb{K}$ any non-zero scalar, then $(cv_1, \ldots, cv_n)$ is a basis. **[4]**

(c) Let $u_1, \ldots, u_r$ be vectors in V. Define the **span** $\langle u_1, \ldots, u_r \rangle$ of $u_1, \ldots, u_r$. **[3]**

(d) Suppose that the list $(u_1, \ldots, u_r)$ is linearly independent but not spanning. Show that there exists a vector $u_{r+1} \in V$ such that $(u_1, \ldots, u_r, u_{r+1})$ is linearly independent. **[4]**
Hint. Choose u_{r+1} to be outside the span $\langle u_1, \ldots, u_r \rangle$ of the original vectors.

(e) Deduce that any linearly independent list in V can be extended to a basis of V. **[3]**

Question 2. [20 marks] This question concerns $n \times n$ matrices over a field $\mathbb{K}$.

(a) In this part only, set $n = 3$. Write down the elementary matrices corresponding to the elementary row operations of (i) adding row 2 to row 1, (ii) interchanging rows 2 and 3, and (iii) multiplying row 1 by the scalar $c \in \mathbb{K}$. **[6]**

(b) Let A be an $n \times n$ matrix. Describe how $\det(A)$ changes when (i) one row of A is added to another, (ii) two rows of A are interchanged, and (iii) one row of A is multiplied by a scalar $c \in \mathbb{K}$. (No justification is required.) **[6]**

(c) Let A and B be non-singular matrices. Prove that $\det(AB) = \det(A)\det(B)$. You may use without proof the fact that any non-singular matrix may be written as the product of elementary matrices.
Hint. Write A as a product of elementary matrices $A = P_t \ldots P_1$. Now compare $\det(A) = \det(P_t \ldots P_1 I)$ with $\det(AB) = \det(P_t \ldots P_1 B)$, where I is the $n \times n$ identity matrix. **[5]**

(d) Suppose that A, B and P are non-singular matrices satisfying $B = P^{-1}AP$. Show that $\det(B) = \det(A)$. **[3]**

Question 3. [20 marks] Suppose α is a linear map on a finite-dimensional vector space V.

(a) Define the **kernel** $\mathrm{Ker}(\alpha)$ and **image** $\mathrm{Im}(\alpha)$ of the linear map α. **[4]**

(b) State, without proof, an identity relating the dimensions of

$$\mathrm{Ker}(\alpha) + \mathrm{Im}(\alpha), \quad \mathrm{Ker}(\alpha) \cap \mathrm{Im}(\alpha), \quad \mathrm{Ker}(\alpha) \quad \text{and} \quad \mathrm{Im}(\alpha).$$

You may assume without proof that $\mathrm{Ker}(\alpha)$ and $\mathrm{Im}(\alpha)$ are subspaces of V. **[3]**

(c) Define what it means for π to be a **projection** on V. **[3]**

(d) Which of the following linear maps on $\mathbb{R}^2$ are projections?

$$\text{(i)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{(ii)} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad \text{(iii)} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \text{(iv)} \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}.$$

No explanation is required. **[4]**

(e) Suppose π is a projection on V. Prove that $\mathrm{Ker}(\pi) \cap \mathrm{Im}(\pi) = \{\mathbf{0}\}$. **[4]**

(f) Deduce that $\dim\big(\mathrm{Ker}(\pi) + \mathrm{Im}(\pi)\big) = \dim(\mathrm{Ker}(\pi)) + \dim(\mathrm{Im}(\pi))$. **[2]**

Question 4. [20 marks] In this question, α is a linear map on a finite-dimensional vector space V, and A is a square matrix representing α relative to some basis.

(a) Define the **characteristic polynomial** $p_A(x)$ of A. **[3]**

(b) State the **Cayley-Hamilton Theorem** as it applies to A. **[3]**

(c) Define the **minimal polynomial** $m_\alpha(x)$ of α. (You are not required to explain why the polynomial exists and is unique.) **[4]**

Recall that the characteristic polynomial of α is defined to be the characteristic polynomial of any matrix A representing it. (The choice of basis is not significant.)

(d) A certain linear map α on $\mathbb{R}^3$ has characteristic polynomial $p_\alpha(x) = (x-1)(x^2+1)$. Is α diagonalisable? Explain your answer. **[3]**

(e) A certain linear map α on $\mathbb{C}^3$ has characteristic polynomial $p_\alpha(x) = (x-1)(x^2+1)$. Is α diagonalisable? Explain your answer. **[3]**

(f) A certain linear map α on $\mathbb{R}^3$ has characteristic polynomial $p_\alpha(x) = (x-1)^3$. Show, by giving two examples, that α may or may not be diagonalisable. **[4]**

Question 5. [20 marks]
In this question, V is a real inner product space, and $\alpha : V \to V$ a linear map on V.

(a) Define the **adjoint** α^* of α. (You are not required to prove existence and uniqueness.) What does it mean for α to be **self-adjoint**? **[4]**

(b) Suppose U and W are subspaces of V. Define what it means for U and W to be **orthogonal**. **[3]**

(c) Define the **orthogonal complement** $U^\perp$ of subspace U. **[3]**

From now on, assume α is self-adjoint.

(d) Suppose v is an eigenvector of α with eigenvalue λ. Let U be the orthogonal complement of $\langle v \rangle$, the one-dimensional subspace spanned by v. Show that $\alpha(u) \in U$ for any $u \in U$. **[5]**

(e) Without giving details, explain how the observation in part (d) is used in the proof of the Spectral Theorem. **[5]**

End of Paper.