Queen Mary
University of London

MTH6139 / MTH6139P: Time Series

Duration: 2 hours

Date and time: May 3, 2016, 10:00–12:00

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators may be used in this examination, but any programming, graph plotting or algebraic facility may not be used. Please state on your answer book the name and type of machine used.
Statistical functions provided by the calculator may be used provided that you state clearly where you have used them.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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Examiner(s): J. Miguez and S. Del Bano Rollin

Question 1 (24 marks). Consider the time series model

$$X_{ij} = m_{ij} + s_j + Y_{ij},$$

where $s_0, s_1, \ldots, s_{d-1}$ is a seasonal component such that $\sum_{j=0}^{d-1} s_j = 0$, m_{ij} is the value of the deterministic trend in the j-th time step ($j = 0, 1, ..., d-1$) of the i-th period ($i = 0, 1, 2, ...$), and $\{Y_{ij}\}$ is a white noise sequence with zero mean and variance σ^2.

(a) Assume that the trend is constant in each period, i.e., $m_{ij} = m_i$ for all j. Show that the estimator

$$\hat{m}_i = \frac{1}{d}\sum_{j=0}^{d-1} X_{ij}$$

is unbiased. **[8]**

(b) Show that the seasonal estimators

$$\hat{s}_j = \frac{1}{n}\sum_{i=0}^{n-1}(X_{ij} - \hat{m}_i), \quad j = 0, 1, \ldots, d-1,$$

satisfy the model constraint $\sum_{j=0}^{d-1} \hat{s}_j = 0$. **[8]**

(c) Assume m_i is linear, i.e., $m_i = a_0 + a_1 i$. Show that both the trend and the seasonality can be eliminated using difference operators. Express the resulting filter in terms of the operator ∇, then express the same filter in terms of the backward shift operator B. **[8]**

Question 2 (26 marks). Consider the time series whose sample autocorrelation function (ACF) and sample partial autocorrelation function (PACF) are given by Figure 1a and 1b, respectively.

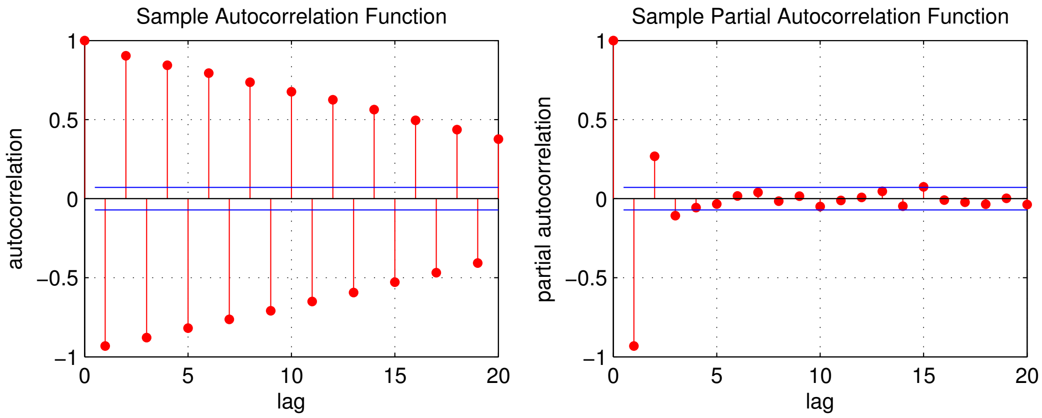
(a) Classify the time series within the ARMA(p, q) family (i.e., estimate p and q). Explain your choice. **[5]**

(b) Consider the time series model $X_t = \phi X_{t-1} + Z_t + \theta Z_{t-1}$, where ϕ and θ are real constant parameters, with $|\phi| < 1$, $\phi \neq \theta$ and $Z_t \sim WN(0, \sigma^2)$. Is it plausible that the sample ACF and PACF in Figure 1a and 1b are generated by a realisation of this model? Justify your answer. **[5]**

(c) Prove that $X_t = \phi X_{t-1} + Z_t + \theta Z_{t-1}$ is invertible when $|\theta| < 1$. **[8]**

(d) Define the PACF.

For the AR(1) model $X_t = \phi X_{t-1} + Z_t$ calculate the partial autocorrelation coefficient for lag $\tau = 1$. **[8]**



(a) Sample ACF for Question 2. (b) Sample PACF for Question 2.

Question 3 (14 marks). Consider the causal AR(2) time series model $X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + Z_t$, where ϕ_1 and ϕ_2 are real constants and $\{Z_t\}$ is a WN(0,σ^2) sequence.

(a) Can the time series model for X_t be inverted? If it can, obtain the inverse model. Otherwise, explain briefly why it is not possible to compute an inverse model. **[4]**

(b) Since $\{Z_t\}$ is white with variance σ^2 and $\{X_t\}$ is causal, for any 1-step-ahead predictor $\hat{X}_t$ the prediction error must be $E[(X_t - \hat{X}_t)^2] \geqslant \sigma^2$. Using this property, prove that the best linear predictor

$$\hat{X}_t = \sum_{i=1}^{t-1} \beta_i X_{t-i}$$

is given by $\beta_1 = \phi_1$, $\beta_2 = \phi_2$ and $\beta_k = 0$ for $k = 3, 4, ...$ **[10]**

Question 4 (22 marks). We have been given time series data $x_1, x_2, ..., x_n$ that can be modelled by means of an AR(2) process

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + Z_t,$$

where $\{Z_t\}$ is white noise with variance σ^2.

(a) Give the Yule-Walker equations that relate the model parameters (ϕ_1, ϕ_2 and σ^2) with the autocorrelation coefficients $\rho(\tau)$. **[8]**

(b) We compute the sample ACVF and the sample ACF of the time series data, which yield estimates

$$\hat{\gamma}(0) = 2, \quad \hat{\rho}(1) = \frac{1}{2}, \quad \hat{\rho}(2) = -\frac{1}{4}.$$

Compute estimates for ϕ_1, ϕ_2 and σ^2. **[6]**

(c) If $n = 100$, give (approximate) 95% confidence intervals for the parameters ϕ_1 and ϕ_2.

Hint: Simply write $\hat{\phi}_1$ and $\hat{\phi}_2$ for the parameter estimates if you have not found numerical values in (b). Also recall that, if U is a normal random variable with 0 mean and unit variance, then $\mathbb{P}(|U| > u_\alpha) = \alpha = 0.95$ for $u_\alpha \approx 1.96$. **[8]**

Question 5 (14 marks). Assume you want to fit an ARIMA(p, d, q) model to represent some time series data $\{x_t\}_{t=1,2,\ldots,n}$ that you have been given.

(a) Give the definition of a general ARIMA(p, d, q) model. [**5**]

(b) Explain briefly how you would fit the parameter d in the model. [**5**]

(c) Assume $d = 2$, $p = 1$ and $q = 2$ have been fitted. Write down an explicit model for the resulting ARIMA(1,2,2) model using polynomials of the backward shift operator B. [**4**]

End of Paper.