Queen Mary University of London

Main Examination period 2019

MTH6132: Relativity

Duration: 2 hours

Student number | | | | | | | | | | **Desk number** | | | |

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Write your solutions in the spaces provided in this exam paper. If you need more paper, ask an invigilator for an additional booklet and attach it to this paper at the end of the exam.

You should attempt ALL questions. Marks available are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiners: H. Bantilan

This page is for marking purposes only.

Do not write on it.

Question	Mark	Comments
1	/ 20	
2	/ 15	
3	/ 10	
4	/ 15	
5	/ 10	
6	/ 10	
7	/ 20	
Total		

Question 1. [20 marks] In an inertial frame F', two events p and q occur simultaneously at a spatial distance of 3 metres apart. In another inertial frame F moving with respect to F', event q occurs later than event p by 10^{-8} seconds.

(a) Draw this situation in a spacetime diagram. For simplicity, let event p be at the origin of both frame F and frame F'. [**10**]

Write your solutions here

(b) Let Δx be the spatial distance between events p and q in the F frame. Write down Δx in metres. Take the speed of light to be $c = 3 \times 10^8$ m/s. **[5]**

Write your solutions here

(c) Write down the velocity v that frame F is moving with respect to frame F', as a fraction of the speed of light c. [**5**]

Write your solutions here

Question 2. [15 marks] Let $ds^2 = g_{ab}dx^a dx^b$ be a metric with components g_{ab} when written in coordinates x^a.

(a) Write down how the metric components g_{ab} transform under a general coordinate transformation. [**5**]

Write your solutions here

(b) The metric of flat two-dimensional space written in Cartesian coordinates $x^a = (x, y)$ is

$$ds^2 = g_{ab}dx^a dx^b = dx^2 + dy^2.$$

Express this metric in polar coordinates r, φ defined by $x = r\cos\varphi$, $y = r\sin\varphi$. **[10]**

Write your solutions here

Question 3. [10 marks] Let $\vec{v}$ be a vector with components v^a when written in coordinates x^a.

(a) Write down the squared norm $||\vec{v}||^2$ with respect to a metric $ds^2 = g_{ab}dx^a dx^b$. [**2**]

Write your solutions here

(b) What are the conditions for $\vec{v}$ to be timelike, spacelike, and null? [**3**]

Write your solutions here

(c) Written in coordinates $x^a = (t,x,y,z)$, a vector $\vec{u}$ has components $u^a = (u^t, u^x, u^y, u^z) = (\alpha, 0, 3\alpha, 0)$, another vector $\vec{v}$ has components $v^a = (v^t, v^x, v^y, v^z) = (3\beta, 0, \beta, 0)$, and the metric is $ds^2 = g_{ab}dx^a dx^b = -dt^2 + dx^2 + dy^2 + dz^2$. Here, α and β are real constants. Determine whether $\vec{u}$ and $\vec{v}$ are timelike, spacelike or null. Under what conditions on α and β would $\vec{w} = \vec{u} + \vec{v}$ be null? **[5]**

Write your solutions here

Question 4. [15 marks]

(a) The metric of the 2-sphere is written in spherical coordinates $x^a = (\theta, \varphi)$ as

$$ds^2 = g_{ab} dx^a dx^b = d\theta^2 + \sin^2\theta d\varphi^2.$$

Consider a curve $x^a(\lambda) = (\theta(\lambda), \varphi(\lambda))$ parametrised by λ. Show that the geodesic equation on the 2-sphere has two components, given by

$$\begin{cases} \dfrac{d^2\theta}{d\lambda^2} - \cos\theta\sin\theta\left(\dfrac{d\varphi}{d\lambda}\right)^2 = 0 \\ \dfrac{d^2\varphi}{d\lambda^2} + 2\dfrac{\cos\theta}{\sin\theta}\left(\dfrac{d\theta}{d\lambda}\right)\left(\dfrac{d\varphi}{d\lambda}\right) = 0. \end{cases}$$

[**10**]

Write your solutions here

Write your solutions here
(continued)

(b) Show that curves with $\varphi = \text{const}$ are geodesics of the 2-sphere.
Remember, you can always parametrise by arc length so that these curves have $d\theta/d\lambda = 1$. [**2**]

Write your solutions here

(c) Show that curves with $\theta = \text{const}$ are geodesics of the 2-sphere if and only if $\theta = \pi/2$. Remember, you can always parametrise by arc length so that these curves have $d\varphi/d\lambda = 1$. [**3**]

Write your solutions here

Question 5. [10 marks] The vacuum Einstein field equations with a cosmological constant Λ are

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 0.$$

where R_{ab} are the components of the Ricci tensor, R is the Ricci scalar, and g_{ab} are the components of the metric of a 4-dimensional spacetime.

(a) What number does the full contraction $g^{ab} g_{ab}$ evaluate to in 4 dimensions? **[2]**

Write your solutions here

(b) Write down the Ricci scalar R in terms of Λ for a 4-dimensional spacetime, by taking the full contraction of the vacuum Einstein field equations with a cosmological constant. [**3**]

Write your solutions here

(c) Using the result in part (b), show that the vacuum Einstein field equations with a cosmological constant can equivalently be written in the much simpler trace-reversed form

$$R_{ab} = \Lambda g_{ab}.$$

[**5**]

Write your solutions here

Question 6. [10 marks] A gravitational wave metric written in coordinates $x^a = (t, x, y, z)$ has components

$$g_{ab} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cos(\omega t - kz),$$

where $h_+, h_\times, \omega, k$ are constants.

(a) Describe the spacetimes with this metric. **[2]**

Write your solutions here

(b) Consider the line $x^a(\lambda) = (t(\lambda), x(\lambda), y(\lambda), z(\lambda))$ parametrised by some λ, from point p with $y = -1$, to point q with $y = 1$, at fixed $t = 0$, $x = 0$, $z = 0$. Parametrise this line by $\lambda = y$, and write down the resulting expression for the tangent vector $dx^a/d\lambda$. [**3**]

Write your solutions here

(c) The arc length of this line is the integral

$$\int_p^q \sqrt{g_{ab}\frac{dx^a}{d\lambda}\frac{dx^b}{d\lambda}}\,d\lambda.$$

Again parametrising by $\lambda = y$ so you can use your expression from part (b), write down the arc length of the line in terms of the constants $h_+, h_\times, \omega, k$ and as a function of t. **[5]**

Write your solutions here

Question 7. [20 marks] The Schwarzschild metric written in static, spherical coordinates $x^a = (t, r, \theta, \varphi)$ is

$$ds^2 = g_{ab}dx^a dx^b = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

where $f(r) = 1 - 2GM/r$ and G, M are constants.

(a) Describe the spacetimes with this metric. **[2]**

Write your solutions here

(b) For any geodesic $x^a(\lambda) = (t(\lambda), r(\lambda), \pi/2, \varphi(\lambda))$ in the Schwarzschild spacetime, it is possible to show that there are two constants E and L so that along the entire geodesic,

$$-E = -f(r)\frac{dt}{d\lambda}$$

$$L = r^2\frac{d\varphi}{d\lambda}.$$

Write down the two properties of the Schwarzschild metric that go into justifying this statement. You do not need to prove the result. [**2**]

Write your solutions here

(c) Write down the norm squared of the tangent vector $dx^a/d\lambda$ for a null geodesic $x^a(\lambda)$. [**2**]

Write your solutions here

(d) For a null geodesic $x^a(\lambda)$ in the Schwarzschild spacetime, using the constants E and L, write down $dr/d\lambda$ as a function only of r **[6]**

Write your solutions here

(e) Write down the norm squared of the tangent vector $dx^a/d\tau$ for a timelike geodesic $x^a(\tau)$ parametrised by proper time τ. [**2**]

Write your solutions here

(f) For a timelike geodesic $x^a(\tau)$ in the Schwarzschild spacetime, using the constants E and L, write down $dr/d\tau$ as a function only of r **[6]**

Write your solutions here

End of Paper – An appendix of 2 pages follows.

Additional work

You are reminded of the following information, which you may use without proof.

- Lower case Latin indices run from 0 to 3.
- The metric tensor of the Minkowski spacetime is η_{ab} such that

$$\mathrm{d}s^2 = \eta_{ab}\mathrm{d}x^a\mathrm{d}x^b = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

- The Lorentz transformations between two frames F and F' in standard configuration are given by

$$x' = \gamma(x - vt), \quad t' = \gamma\left(t - \frac{vx}{c^2}\right), \quad y' = y, \quad z' = z$$

  where

$$\gamma = \frac{1}{\sqrt{1-(v^2/c^2)}}$$

  and F' is moving with speed v relative to F.

- The covariant derivative of a covariant vector is given by

$$\nabla_a V_b = \partial_a V_b - \Gamma^f{}_{ba} V_f.$$

- The covariant derivative of a contravariant vector is given by

$$\nabla_a V^b = \partial_a V^b + \Gamma^b{}_{af} V^f.$$

- The metric tensor satisfies:

$$g_{ab} g^{bc} = \delta_a{}^c.$$

- Christoffel symbols (connection):

$$\Gamma^c{}_{ab} = \tfrac{1}{2} g^{cd} \left(\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}\right).$$

- The Riemann curvature tensor:

$$R^a{}_{bcd} = \partial_c \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{bc} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc}.$$

- Euler–Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial L}{\partial \dot{x}^c}\right) - \frac{\partial L}{\partial x^c} = 0$$

- Geodesic equations:

$$\ddot{x}^a + \Gamma^a{}_{bc}\dot{x}^b\dot{x}^c = 0.$$

End of Appendix.