Queen Mary
University of London

B. Sc. Examination by course unit 2015

MTH6128: Number Theory

Duration: 2 hours

Date and time: 22 May 2015, 10:00 to 12:00

You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and **cross through any work that is not to be assessed**.

Possession of unauthorised material at any time when under examination conditions is an assessment offence and can lead to expulsion from QMUL. Check now to ensure you do not have any notes, mobile phones, smartwatches or unauthorised electronic devices on your person. If you do, raise your hand and give them to an invigilator immediately. It is also an offence to have any writing of any kind on your person, including on your body. If you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms it shall be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. A mobile phone that causes a disruption in the exam is also an assessment offence.

Exam papers must not be removed from the examination room.

Examiner(s): X. Li

Question 1. (a) What is an *algebraic number*? What is an *algebraic integer*? What is a *transcendental number*? [**4**]

(b) What is a *quadratic number*? What is a *quadratic integer*? [**4**]

(c) Give an example of an algebraic number which is not an algebraic integer. Explain why the number you gave has the desired properties. [**3**]

(d) Which of the following numbers are algebraic integers? Explain, stating precisely all theorems you use.

(i) $\dfrac{3+\sqrt{5}}{2}+1$; [**2**]

(ii) $\dfrac{1}{2}\sqrt{41}-\dfrac{3}{2}$. [**2**]

Question 2. (a) Use the Euclidean algorithm to find the continued fraction expansion of $\dfrac{261}{128}$. [**6**]

(b) Find the continued fraction for $\dfrac{1+\sqrt{37}}{3}$. [**8**]

(c) Calculate the value of the infinite continued fraction $[2;\overline{3,1}]$. [**8**]

Question 3. (a) You are given that

$$[9;\overline{2,3,3,2,18}]$$

is the continued fraction for $\sqrt{89}$. Using this, find positive integers x and y such that $x^2+y^2=89$. [**6**]

(b) You are given that

$$[7;\overline{3,1,1,3,14}]$$

is the continued fraction for $\sqrt{53}$. Using this, find all the integer solutions of the equation

$$x^2-53y^2=\pm 1.$$

Explain why you have found ALL the integer solutions. [**8**]

Question 4. (a) Suppose n is a natural number with $n \equiv 3 \pmod 4$. Show that we cannot find integers x and y with $x^2 + y^2 = n$. [**4**]

(b) Suppose n is a natural number with $n \equiv 7 \pmod 8$. Show that we cannot find integers x, y and z with $x^2 + y^2 + z^2 = n$. [**4**]

Question 5. (a) Let p be an odd prime. What is a *quadratic residue* (mod p)? What is a *quadratic non-residue* (mod p)? [**4**]

(b) Let p be an odd prime, and let a be an integer. Define the *Legendre symbol* $\left(\dfrac{a}{p}\right)$. [**3**]

(c) Calculate the value of $\left(\dfrac{18}{59}\right)$. You should state clearly any rules for computing Legendre symbols that you use, but are not required to prove them. [**6**]

(d) Find an integer a such that $\left(\dfrac{a}{61}\right) = -1$. Explain why the integer you found has the desired property. You should state clearly any rules for computing Legendre symbols that you use, but are not required to prove them. [**6**]

(e) Show that there are infinitely many prime numbers p with $\left(\dfrac{-1}{p}\right) = -1$. State precisely all results from lectures you use in the proof. [**6**]

Question 6. (a) What is a *quadratic form* over the integers? [**1**]

(b) Give an example of a quadratic form which is positive definite. Explain why the example you gave has the desired property. [**2**]

(c) Give an example of a quadratic form which is degenerate. Explain why the example you gave has the desired property. [**2**]

(d) What is meant by saying that two quadratic forms are *equivalent*? [**2**]

(e) Find a reduced positive definite quadratic form which is equivalent to [**3**]

$$3x^2 - 4xy + 2y^2.$$

(f) Let f and g be two quadratic forms. Prove that f and g are equivalent if and only if $-f$ and $-g$ are equivalent. [**6**]

End of Paper.