Queen Mary
University of London

B. Sc. Examination by course unit 2014

MTH6128 Number Theory

Duration: 2 hours

Date and time: 21 May 2014, 10:00 to 12:00

You should attempt all questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Important note: the Academic Regulations state that possession of unauthorized material at any time by a student who is under examination conditions is an assessment offence and can lead to expulsion from QMUL.

Please check now to ensure you do not have any notes, mobile phones or unauthorised electronic devices on your person. If you have any, then please raise your hand and give them to an invigilator immediately. Please be aware that if you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms, it will be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. Disruption caused by mobile phones is also an examination offence.

Exam papers must not be removed from the examination room.

Examiner(s): X. Li

Question 1 (a) What is an *algebraic number*? What is an *algebraic integer*? What is a *transcendental number*? [4]

(b) Which of the following numbers are algebraic integers? Explain, stating precisely all theorems you use.

(i) $\dfrac{3+\sqrt{5}}{2}+\dfrac{1}{5}$; [4]

(ii) $\dfrac{1}{2}\sqrt{21}-\dfrac{1}{2}$.

Question 2 (a) Use the Euclidean algorithm to find the greatest common divisor of 263 and 108. [4]

(b) Use your working from (a) to find a continued fraction expansion of $\dfrac{263}{108}$. [4]

Question 3 (a) Let $a_0, a_1, a_2, \ldots$ be a sequence of integers, with $a_n > 0$ for all $n \geq 1$. How is the *value of the infinite continued fraction* $[a_0; a_1, a_2, \ldots]$ defined? [2]

(b) Calculate the value of the infinite continued fraction [5]

$$[1; \overline{1, 2}] = [1; 1, 2, 1, 2, 1, 2, \ldots].$$

(c) Show that the value of the periodic continued fraction

$$[a_0; a_1, \ldots, a_m, \overline{a_{m+1}, \ldots, a_{m+k}}]$$

is a quadratic number. [7]

Question 4 (a) Explain how to use the continued fraction for $\sqrt{p}$ (where p is a prime congruent to 1 modulo 4) to find positive integers x and y satisfying the equation $x^2 + y^2 = p$. [4]

(b) Find the continued fraction for $\sqrt{73}$. [8]

(c) Using parts (a) and (b), find positive integers x and y such that $x^2 + y^2 = 73$. [4]

(d) Find all the integer solutions of the equation

$$x^2 + y^2 = 73.$$

Explain why you have found ALL the integer solutions. [3]

(e) Find all the integer solutions of the equation

$$x^2 - 73y^2 = \pm 1.$$

Explain why you have found ALL the integer solutions. [8]

Question 5 (a) Let p be an odd prime. Define the *Legendre symbol* $\left(\frac{a}{p}\right)$ for any integer a. [3]

(b) Calculate the value of $\left(\frac{51}{61}\right)$. You should state clearly any rules for computing Legendre symbols that you use, but are not required to prove them. [6]

(c) Let p be an odd prime. Show that $\left(\frac{-3}{p}\right) = +1$ if and only if $p \equiv 1 \pmod 6$. [8]

(d) Prove that any prime greater than 3 is congruent to 1 or -1 modulo 6. [2]

(e) Show that there are infinitely many prime numbers p with $\left(\frac{-3}{p}\right) = -1$. [8]

Question 6 (a) What is a quadratic form over the integers? Define the *discriminant* of a quadratic form over the integers. [2]

(b) In each of the following cases, state whether the quadratic form is positive definite, negative definite, indefinite, or degenerate:

(i) $-2x^2 + 3xy - 4y^2$;

(ii) $-5x^2 - 4xy + 3y^2$. [2]

(c) What is meant by saying that a positive definite quadratic form is *reduced*? When are two reduced positive definite quadratic forms equivalent? [2]

(d) Find a reduced positive definite quadratic form which is equivalent to $5x^2 - 4xy + 2y^2$. [2]

(e) Find a reduced positive definite quadratic form which is equivalent to $31x^2 - 10xy + y^2$. [2]

(f) Find an integer a such that the quadratic forms $x^2 + y^2$ and $ax^2 - 20xy + y^2$ are equivalent. Prove that the integer you have found has the desired property. [6]

End of Paper