Queen Mary University of London

**Main Examination period 2023 – May/June – Semester B**

# MTH6127 / MTH6127P: Metric Spaces and Topology

**Duration: 2 hours**

The exam is intended to be completed within **2 hours**. However, you will have a period of **4 hours** to complete the exam and submit your solutions.

**You should attempt ALL questions. Marks available are shown next to the questions.**

All work should be **handwritten** and should **include your student number**. Only one attempt is allowed – **once you have submitted your work, it is final**.

In completing this assessment:

- You may use books and notes.
- You may use calculators and computers, but you must show your working for any calculations you do.
- You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.
- You must not seek or obtain help from anyone else.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;

**Examiners: M. Farber, M. Shamis**

In this examination $\mathbb{R}$ denotes the set of real numbers, $\mathbb{Q}$ denotes the set of rational numbers and $\mathbb{Z}$ denotes the set of integers.

**Question 1 [10 marks].**

(a) Show that any open set $U \subset X$ in a metric space is a union of a family of open balls $B(c; r)$ having radii $r < 0.1$. **[5]**

(b) In a metric space $(X, d)$ the metric $d : X \times X \to \mathbb{R}$ satisfies $d(x, y) \geq 0.2$ for all $x, y \in X$ with $x \neq y$. Describe the open subsets of $X$. **[5]**

**Question 2 [5 marks].** For points $x, y, z$ in a metric space $(X, d)$ it is known that $d(x, y) = 1$ and $d(y, z) = 2$. Based on this information and using general properties of metric spaces describe the possible range for $d(x, z)$. Justify your answer. **[5]**

**Question 3 [10 marks].**

(a) When do we say that a sequence $\{x_n\}_{n \geqslant 1}$ of points in a metric space $X$ *converges*? **[2]**

(b) Give the definition of *a Cauchy sequence* in a metric space $(X, d)$. **[2]**

(c) Prove that a Cauchy sequence converges if it has a convergent subsequence. **[6]**

**Question 4 [10 marks].**

(a) Let $X$ be a metric space and let $A \subseteq X$ be a subset which is not closed. Show that $A$ is not complete with respect to the induced metric. **[2]**

(b) Which of the following subsets of $\mathbb{R}$ are complete when considered as subspaces of $\mathbb{R}$ with the usual metric? Briefly explain your answer.

(i) $(1, \infty)$, **[2]**

(ii) $[1, \infty)$, **[2]**

(iii) $\{n^{-2};\, n = 1, 2, \dots\}$, **[2]**

(iv) $\{n^{-2}; n = 1, 2, \dots\} \cup \{0\}$. **[2]**

**Question 5 [10 marks].**

(a) Describe the finite complement topology on a set $X$. Show that this topology is Hausdorff if and only if the set $X$ is finite. **[7]**

(b) Show that any metric space equipped with the topology induced by the metric is Hausdorff. **[2]**

 

(c) Is $(0, 1]$ equipped with the topology induced from the standard topology of $\mathbb{R}$ Hausdorff? Briefly explain your answer. **[1]**

**Question 6 [25 marks].**

(a) Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Is it possible that $f([0, 1]) = (1, 2)$? Justify your answer. **[6]**

(b) Describe a homeomorphism $\phi : (-\pi, \pi) \to (1, 2)$ and its inverse. . **[4]**

(c) Are the intervals $(-\pi, \pi)$ and $(1, 2)$ equipped with the standard metric isometric? Justify your answer. **[6]**

(d) Which of the following subsets of the real line $\mathbb{R}$ are compact; briefly explain your answer:

(i) $[-\pi, \pi]$; **[1]**

(ii) $(2, 5)$; **[2]**

(iii) $[6, \infty)$; **[2]**

(iv) $\mathbb{R}$; **[2]**

(v) $\{n^{-1}; n = 1, 2, \dots\}$. **[2]**

**Question 7 [10 marks].**

(a) Consider the map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \frac{1}{2}(x + 1)$. Is this map a contraction? Justify your answer. **[3]**

(b) Show that $f$ maps the interval $(1, 2]$ to itself. **[1]**

(c) Is the contraction mapping theorem applicable to the restriction map $f : (1, 2] \to (1, 2]$? Justify your answer. **[3]**

(d) Does the map $f : (1, 2] \to (1, 2]$ have a fixed point $x \in (1, 2]$? **[3]**

**Question 8 [20 marks].**

(a) Consider $\mathbb{R}^2$ with the $d_1$-metric, i.e. $d_1(v, v') = |x - x'| + |y - y'|$ where $v = (x, y)$ and $v' = (x', y')$. Is this metric space complete? Justify your answer. **[4]**

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(v) = (\frac{1}{2}y, \frac{1}{2}(x + 1))$ where $v = (x, y)$. Show that $f$ is a contraction with respect to $d_1$-metric. **[10]**

(c) Find the fixed point of $f$. **[6]**

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**End of Paper.**