Queen Mary
University of London

Main Examination period 2022 – May/June – Semester B

MTH6127: Metric Spaces and Topology

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment:

- **You may use books and notes.**
- **You may use calculators and computers, but you must show your working for any calculations you do.**
- **You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.**
- **You must not seek or obtain help from anyone else.**

All work should be **handwritten** and should **include your student number**.

The exam is available for a period of **24 hours**. Upon accessing the exam, you will have **3 hours** in which to complete and submit this assessment.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

Please try to upload your work well before the end of the submission window, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

Examiners: M. Godazgar, R. Buzano

Unless explicitly stated, you may use, without proof, any theorem or claim proved in the full lecture notes or any exercise contained therein. However, you must make clear which result you are using in your answer.
In this examination $\mathbb{R}$ stands for the set of real numbers, $\mathbb{R}^+$ stands for positive, non-zero real numbers and $\mathbb{N} \equiv \{1, 2, 3, \ldots\}$ stands for the set of natural numbers.

Question 1 [25 marks].

(a) *"Given a metric space (X, d), a sequence $(x_n)_{n=1}^{\infty}$ in X is Cauchy if and only if $\forall \epsilon > 0, \exists N \in \mathbb{N}\ \forall m, n,\ |m - n| < N,\ d(x_m, x_n) < \epsilon$."*

Is the above definition correct? If not, identify the problem and fix it. **[4]**

Consider the metric space $(C^0([0,1]), d_{L^1})$ of continuous real-valued functions on $[0, 1]$ with the L^1 metric defined for $f, g \in C^0([0,1])$ by

$$d_{L^1}(f, g) = \int_0^1 |f(x) - g(x)|\, dx.$$

(b) Prove that the following sequence $(f_n)_{n=1}^{\infty} \in C^0([0,1])$ is Cauchy:

$$f_n(x) = \begin{cases} 0 & x \in [0, \frac{1}{2} - \frac{1}{n}] \\ n(x + \frac{1}{n} - \frac{1}{2}) & x \in [\frac{1}{2} - \frac{1}{n}, \frac{1}{2}] \\ 1 & x \in [\frac{1}{2}, 1] \end{cases}.$$

[12]

(c) Show that (f_n) cannot converge to a continuous function. **[7]**

(d) What does this say about the metric space $(C^0([0,1]), d_{L^1})$? **[2]**

Question 2 [25 marks]. Given a vector space V, let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space.

(a) *"Any inner product space induces a normed space with norm $||x|| = \langle x, x\rangle^2$."*

Is the above assertion correct? If not, identify the problem and fix it. [**5**]

(b) Given non-zero vectors $x, y \in V$, identifying relevant properties of normed and inner product spaces, show that for rescaled vectors $x' = \frac{x}{||x||^2}, y' = \frac{y}{||y||^2}$,

$$||x' - y'||^2 = \frac{1}{||x||^2} - \frac{2\langle x, y\rangle}{||x||^2||y||^2} + \frac{1}{||y||^2}.$$ [**7**]

(c) Starting from the triangle inequality for appropriately defined vectors x', y', z', derive the following inequality:

$$||x - y||\ ||z|| \le ||y - z||\ ||x|| + ||z - x||\ ||y||.$$ [**13**]

Question 3 [25 marks]. Let (X, τ) be a topological space. Let Y be a subset of X.

(a) *"The relative topology τ_Y on a subset Y is equal to $\tau \cup Y$"*

Is the above definition correct? If not, identify the problem and fix it. [**4**]

(b) *"Y is a connected subset if and only if there exist relatively open sets $A, B \subseteq Y$ such that $Y = A \cup B$ and $A \cap B \neq \emptyset$."*

Is the above definition correct? If not, identify the problem and fix it. [**4**]

(c) Consider connected subsets $A_i \subseteq X$, $(i = 1, \ldots, N \in \mathbb{N})$ such that

$$\bigcap_{i=1}^{N} A_i \neq \emptyset.$$

Prove that $\bigcup_{i=1}^{N} A_i$ is connected. [**10**]

(d) Let $X = \bigcup_{i=1}^{N} A_i$ with $N \in \mathbb{N}$, where each A_i is compact. Prove that X is compact. [**7**]

Question 4 [25 marks].

(a) *"Let (X,d) be a compact metric space and $f : X \to X$ be a contraction mapping. Then f has a unique fixed point."*

Is the above assertion correct? If not, identify the problem and fix it. [**4**]

(b) Consider $\mathbb{R}^2$ with the Euclidean metric. Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(x,y) = \left(\frac{1}{2}x + 1, \frac{1}{3}(y-1)\right), \qquad x, y \in \mathbb{R}^2.$$

Show that f is a contraction mapping. [**6**]

(c) Find a fixed point of f. Is this fixed point unique? Justify your answer. [**7**]

(d) Given an arbitrary metric d on a non-empty set X, for any $x, y \in X$ consider

$$\tilde{d}(x,y) = d(x,y)^2, \qquad \hat{d}(x,y) = k\,d(x,y) \quad (k \in \mathbb{R}).$$

Do $\tilde{d}$ and $\hat{d}$ define metrics on X? [**8**]

End of Paper.