Queen Mary University of London

Main Examination period 2021 – May/June – Semester B
Online Alternative Assessments

MTH6127: Metric Spaces and Topology

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment:

- **You may use books and notes.**
- **You may use calculators and computers, but you must show your working for any calculations you do.**
- **You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.**
- **You must not seek or obtain help from anyone else.**

All work should be **handwritten** and should **include your student number**.

You have **24 hours** to complete and submit this assessment. When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

You are expected to spend about 2 hours to complete the assessment, plus the time taken to scan and upload your work. Please try to upload your work well before the end of the submission window, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

Examiners: M. Godazgar, R. Buzano

Unless explicitly stated, you may use, without proof, any theorem or claim proved in the full lecture notes or any exercise contained therein. However, you must make clear which result you are using in your answer.
In this examination $\mathbb{R}$ stands for the set of real numbers, $\mathbb{R}^+$ stands for positive, non-zero real numbers and $\mathbb{N} \equiv \{1, 2, 3, \ldots\}$ stands for the set of natural numbers.

Question 1 [25 marks].

(a) Let (X, d) be a metric space. Consider a subset $Y \subseteq X$. Under what conditions, if any, can one define a metric space on Y? Justify your answer. **[4]**

(b) Let $X = \{a, b, c\}$ with

$$d(a,b) = d(b,a) = n,$$
$$d(b,c) = d(c,b) = m > n,$$
$$d(a,c) = d(c,a) = r,$$
$$d(a,a) = d(b,b) = d(c,c) = 0.$$

Prove that there exist choices of $m, n, r \in \mathbb{R}$ such that (X, d) defines a metric space. **[11]**

(c) (i) What is the topology induced by the metric space found in part (b) above? Justify your answer. **[4]**

(ii) Is this topology connected? **[3]**

(d) Is it possible to define a topology on the set $X = \{a, b, c\}$ such that $\{a\}$ is open and not closed, while $\{b\}$ and $\{c\}$ are closed and not open? **[3]**

Question 2 [25 marks]. Let (X, d) be a metric space.

(a) *"The subset $\Omega \subseteq X$ is called open iff $\exists\, x \in \Omega,\ r > 0$ such that $B_r(x) \subseteq \Omega$."*

Is the above definition correct? If not, identify the problem and fix it. **[4]**

(b) Let U, V be open subsets of X. Prove from the appropriate definition that $U \cap V$ is open. **[5]**

(c) (i) Assuming the result of b) above, prove that given a finite collection $U_1, \ldots, U_N$ open,

$$\bigcap_{i=1}^{N} U_i \quad \text{is open.}$$ **[5]**

(ii) Why can't this method be used to prove that for countably many open sets U_i $(i \in \mathbb{N})$

$$\bigcap_{i=1}^{\infty} U_i \quad \text{is open?}$$ **[3]**

(d) Under what conditions is

$$d'(x,y) = a\, d(x,y) + b, \qquad a, b \in \mathbb{R}$$

a metric? Under such conditions, argue that the metrics d and d' are equivalent. **[8]**

Question 3 [25 marks]. Let (X, d) be a metric space.

(a) *"A function $f : X \to X$ is called a contraction mapping iff $\exists$ a real number $\alpha \leq 1$ such that*

$$d\left(f(x), f(y)\right) \leq \alpha\, d(x, y) \quad \forall x, y \in X."$$

Is the above definition correct? If not, identify the problem and fix it. **[4]**

(b) Consider $X = \mathbb{R}^2$ with metric

$$d(u, \tilde{u}) = \tfrac{1}{2}|x - \tilde{x}| + |y - \tilde{y}|, \qquad u = (x, y),\ \tilde{u} = (\tilde{x}, \tilde{y}). \tag{1}$$

Is this metric space complete? Justify your answer. **[6]**

[You may assume that $(\mathbb{R}, d(x, y) = |x - y|)$ is complete. You do not need to verify that d defines a metric.]

(c) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(u) = \left(\tfrac{1}{2}y, \tfrac{1}{3}(x - 1)\right), \qquad u = (x, y).$$

Show that f defines a contraction with respect to the metric defined in (1). **[9]**

(d) Find a fixed point of f. Is it unique? Justify your answer. **[6]**

Question 4 [25 marks].

(a) Consider the sequence of functions

$$f_n(x) = \sin(nx), \qquad x \in [0, \pi].$$

Does this sequence converge in $\mathcal{B}([0, \pi])$, the space of bounded, continuous functions with respect to the sup-metric? **[4]**

(b) *"Given topological spaces (X, τ_X) and (Y, τ_Y), a function $f : X \to Y$ is called continuous iff*

$$\forall \Omega \in \tau_X, \quad f(\Omega) \in \tau_Y."$$

Is the above definition correct? If not, identify the problem and fix it. **[4]**

(c) In $\mathbb{R}^n$, for $a, b \in \mathbb{R}^+$, consider the subset

$$Y = \{x \in \mathbb{R}^n : a < ||x||^2 < b\},$$

where $|| \cdot ||$ corresponds to the Euclidean norm on $\mathbb{R}^n$.

(i) Show that Y is open. **[6]**

(ii) Deduce using the above result and the definition of continuity that the function $f : \mathbb{R}^n \to \mathbb{R}^+$ (with Euclidean metrics), $f(x) = ||x||^2$ is continuous. **[4]**

(d) Using the above results, show that the subset $K = \{x \in \mathbb{R}^n : ||x||^2 = 1\}$ is closed in $\mathbb{R}^n$. **[3]**

(e) Prove that the set K defined in part (d) is compact. **[4]**

End of Paper.