Queen Mary
University of London

Main Examination period 2017

MTH6127 / MTH6127P
Metric Spaces and Topology

Duration: 2 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks available are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

Possession of unauthorised material at any time when under examination conditions is an assessment offence and can lead to expulsion from QMUL. Check now to ensure you do not have any notes, mobile phones, smartwatches or unauthorised electronic devices on your person. If you do, raise your hand and give them to an invigilator immediately.

It is also an offence to have any writing of any kind on your person, including on your body. If you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms, it shall be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. A mobile phone that causes a disruption in the exam is also an assessment offence.

Exam papers must not be removed from the examination room.

Examiners: M. Farber

In this examination the symbol $\mathbb{R}$ denotes the set of real numbers.

Question 1. [6 marks] Let X be a set and let $d : X \times X \to \mathbb{R}$ be a function.

(a) State the three properties (axioms) that d must satisfy to be a metric on X. [**2**]

(b) Let X be a set and let $d : X \times X \to \mathbb{R}$ be given by

$$d(x,y) = \begin{cases} 0, & \text{if} \quad x = y, \\ 1, & \text{if} \quad x \neq y. \end{cases}$$

Prove that d is a metric on X. [**4**]

Question 2. [20 marks]

(a) Let $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given by

$$d(x,y) = |x - y|$$

for $x, y \in \mathbb{R}$. Prove that d is a metric on $\mathbb{R}$. [**4**]

(b) Let $d : X \times X \to \mathbb{R}$ be a metric on X. Define $d' : X \times X \to \mathbb{R}$ by

$$d'(x,y) = \sqrt{d(x,y)}.$$

Prove that d' is a metric on X. [**6**]

Hint: You may use the inequality $\sqrt{a+b} \leqslant \sqrt{a} + \sqrt{b}$ where $a \geqslant 0$ and $b \geqslant 0$.

(c) Is the function $d' : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$,

$$d'(x,y) = |x - y|^{1/4}, \quad x, y \in \mathbb{R}$$

a metric on the real line? Justify your answer. [**5**]

Hint: You may use the result of Question 2, part (b).

(d) Let $\tilde{d} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given by

$$\tilde{d}(x,y) = |x - y|^2, \quad x, y \in \mathbb{R}.$$

Is $\tilde{d}$ a metric on the real line? Justify your answer. [**5**]

Hint: Compute $\tilde{d}(0,1)$, $\tilde{d}(1,2)$ and $\tilde{d}(0,2)$.

Question 3. [20 marks]

(a) When do we say that a sequence of points $\{x_n\}$ in a metric space (X,d) **converges**? [2]

(b) Give the definition of **a Cauchy sequence** in a metric space (X,d). Show that any convergent sequence is a Cauchy sequence. [4]

(c) Define what is meant for a metric space (X,d) to be **complete**. Give an example of a metric space which is not complete. [4]

(d) Let X be a set and let $d : X \times X \to \mathbb{R}$ be given by

$$d(x,y) = \begin{cases} 0, & \text{if} \quad x = y, \\ 1, & \text{if} \quad x \neq y. \end{cases}$$

Is (X,d) complete? Justify your answer. [4]

(e) Which of the following subsets of $\mathbb{R}$ are complete when considered as subspaces of $\mathbb{R}$ equipped with the usual metric? Briefly explain your answer.

(i) $\{n^{-2}; n = 1,2,\dots\}$, [3]

(ii) $\{n^{-2}; n = 1,2,\dots\} \cup \{0\}$. [3]

Question 4. [10 marks]

(a) Define what it means for a topological space X to be **Hausdorff**? [2]

(b) Give an example of a topological space which is not Hausdorff. [2]

(c) Give an example of a sequence of points $\{x_n\}$ in a topological space X converging to several distinct points. [3]

(d) Show that in a Hausdorff topological space X a sequence of points $\{x_n\}$ can converge to at most one point. [3]

Question 5. [26 marks]

(a) Define what it means that a topological space is **compact**? **[3]**

(b) Prove that any compact subset $A \subseteq X$ of a metric space (X,d) is **bounded**, i.e. there exists $N > 0$ such that $d(x,y) \leqslant N$ for all $x, y \in A$. **[5]**

(c) Prove that any compact subset of a Hausdorff topological space is closed. **[5]**

(d) State the criterion of compactness for subsets of the Euclidean space $\mathbb{R}^n$. **[3]**

(e) Which of the following subsets of the real line $\mathbb{R}$ are compact? Briefly explain your answer.

(i) $[0,1]$; **[2]**

(ii) $(0,1)$; **[2]**

(iii) $[0,\infty)$; **[2]**

(iv) $\mathbb{R}$; **[2]**

(v) $\{n^{-1}; n = 1,2,\dots\}$. **[2]**

Question 6. [18 marks]

(a) Let (X,d) be a metric space. Define what it means that a mapping $T : X \to X$ is **a contraction**? **[3]**

(b) State the contraction mapping theorem. No proof is required. **[3]**

(c) State whether the space of continuous functions $C[a,b]$ with the sup-metric is complete. No proof is required. **[1]**

(d) Consider the map

$$T : C[0,1/2] \to C[0,1/2]$$

given by the formula

$$T(f)(t) = tf(t) + t, \quad f \in C[0,1/2], \quad t \in [0,1/2].$$

Prove that T is a contraction mapping. **[6]**

(e) Find $f \in C[0,1/2]$ such that $T(f) = f$. **[5]**

End of Paper.