Queen Mary
University of London

B. Sc. Examination by course unit 2014

MTH6121: Introduction to Mathematical Finance

Duration: 2 hours

Date and time: 28 April 2014, 14:30–16:30

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt all questions. Marks awarded are shown next to the questions.

Calculators ARE permitted in this examination. The unauthorised use of material stored in pre-programmable memory constitutes an examination offence. Please state on your answer book the name and type of machine used.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Important note: the Academic Regulations state that possession of unauthorised material at any time by a student who is under examination conditions is an assessment offence and can lead to expulsion from QMUL.

Please check now to ensure you do not have any notes, mobile phones or unauthorised electronic devices on your person. If you have any, then please raise your hand and give them to an invigilator immediately. Please be aware that if you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms, it will be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. Disruption caused by mobile phones is also an examination offence.

Exam papers must not be removed from the examination room.

Examiner(s): I. Goldsheid

Question 1 Let X be a continuous random variable with probability density function $f_X(x)$.

(a) What is the definition of the probability density function? [2]

(b) What is the definition of the n-th moment of X, where n is a positive integer? How can one compute the n-th moment of X in terms of its probability density function? [2]

(c) Suppose now that X is an exponential random variable with

$$f_X(x) = \begin{cases} e^{-x} & \text{if } x \geq 0; \\ 0 & \text{if } x < 0. \end{cases}$$

Determine the characteristic function G_X of this random variable. [3]

(d) Using (c), or otherwise, determine all moments of X. [5]

Question 2

(a) Define what is a lognormal random variable with parameters (μ, σ^2). [2]

(b) Let $X_1, X_2, \ldots, X_n$ be independent random variables with $X_i \sim \text{LogNormal}(\mu_i, \sigma_i^2)$ and suppose that $a_1, a_2, \ldots, a_n$ are real constants. Show that

$$\prod_{i=1}^{n} X_i^{a_i} \sim \text{LogNormal}(m_n, s_n^2)$$

for appropriate parameters m_n and s_n and give explicit formulae for m_n and s_n. Here, $\prod_{i=1}^{n} X_i^{a_i}$ denotes the product $X_1^{a_1} X_2^{a_2} \cdots X_n^{a_n}$. [7]

Question 3

(a) Give the definition of a Wiener process. [5]

(b) Define what is a Brownian motion with drift parameter μ and volatility parameter σ. [3]

(c) Suppose that the price of a share evolves according to a Geometric Brownian motion with drift parameter $\mu = 0.004$ and volatility parameter $\sigma = 0.4$. What is the probability that the price of the share at time $t = 4$ is higher than at time $t = 0$ and at time $t = 8$ is lower than at time $t = 4$? [7]

Question 4 A five year coupon bond costs £1000 and pays its holder £50 at the end of the following ten six-months periods with a final additional payment of £1000. Assuming that interest is compounded monthly, find the present value V of the cash flow generated by this bond if the interest rate is 6%.

If the interest rate increases, will the present value V become larger or smaller? Explain your answer. [8]

Question 5 Explain what is meant by the following financial instruments and describe the corresponding cashflows:

(a) a fixed interest security; [5]

(b) an interest only loan. [5]

Question 6

(a) Define what is an arbitrage opportunity. [2]

(b) State the Law of One Price. [3]

(c) Using the Law of One Price or otherwise prove the following proposition.

Proposition Suppose that interest is compounded continuously at nominal rate r. Consider a forward contract at time 0 to pay an amount F for a unit of stock to be delivered and paid for at time t. In order to have no arbitrage on stock having price S at time 0 we must have

$$F = Se^{rt}\,.$$

[8]

Question 7 You are reminded that the arbitrage theorem is related to a situation when investors bet on n wagers and there are m possible outcomes of the experiment characterized by return functions $r_i(\cdot)$. It states:

Theorem *Exactly one of the two following statements is true:*
Either
(i) There is a probability vector $\mathbf{p} = (p_1, p_2, ..., p_m)$ *such that*

$$\sum_{j=1}^{m} p_j r_i(j) = 0 \quad \textit{for all } i = 1, ..., n. \tag{1}$$

Or
(ii) There is a betting strategy $\mathbf{x} = (x_1, x_2, ..., x_n)$ *for which*

$$\sum_{i=1}^{n} x_i r_i(j) > 0 \quad \textit{for all } \; j = 1, ..., m. \tag{2}$$

(a) Explain briefly the meaning of the numbers x_i and $r_i(j)$. [3]

(b) Prove that if (1) holds for all i, $1 \le i \le n$, then it is impossible to have strict inequalities (2) for all j, $1 \le j \le m$. [6]

(c) Consider an experiment with n possible outcomes, and n possible bets (horse racing would be a typical real life example of such a situation). The betting conditions are as follows: if you bet x on outcome i then you get back xo_i if the result is i, and you lose your bet if the result is not i. It is also given that all $o_i \ge 0$. The corresponding return function is

$$r_i(j) = \begin{cases} o_i & \text{if } i = j; \\ -1 & \text{if } i \neq j. \end{cases}$$

Prove that for there not to be an arbitrage opportunity the following relation should hold:

$$\sum_{i=1}^{n} \frac{1}{o_i + 1} = 1.$$

[8]

Question 8

(a) Explain what is a call option. [3]

(b) Consider the Black–Scholes Model with drift parameter μ and volatility parameter σ. Show that the probability that a call option with strike price K and expiration time t on stock with current price S will be exercised is

$$1 - \Phi\left(\frac{\ln\frac{K}{S} - \mu t}{\sigma\sqrt{t}}\right),$$

where Φ is the cumulative distribution function of a standard normal random variable. [5]

(c) In the Black-Scholes model suppose that a share is currently selling at a price of 20, the nominal interest rate is 6% and the volatility is 0.40. Find the no-arbitrage cost C of a call option if its expiration time is three months from now and the strike price is 24.

Suppose now that the interest rate decreases while all other parameters remain the same. Will the cost of the option become smaller or larger? State (do not prove) the property of the Black–Scholes formula which provides the answer. [8]

End of Paper — A one-page appendix follows