Queen Mary University of London

Main Examination period 2017

MTH6109 / MTH6109P: Combinatorics

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Examiners: R. A. Wilson, M. Fayers, B. Noohi

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Question 1. [20 marks] Give, with justification, a simple formula for the each of the following. *(You need not evaluate factorials or large powers.)*

(a) The number of sequences of 8 letters (with repetitions allowed) from an alphabet of 26 letters. [**5**]

(b) The number of partitions of a set of size 28 into 7 disjoint subsets of size 4. [**5**]

(c) The number of ways of distributing 12 identical bottles of lemonade to 5 children, such that each child gets at least one bottle. [**5**]

(d) The number of ways of distributing 12 identical bottles of lemonade to 5 children, such that each child gets at least one bottle, but no child gets more than three bottles. [**5**]

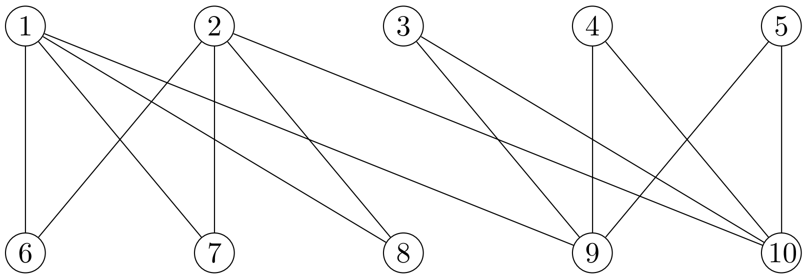
Question 2. [12 marks] Suppose G is a graph, with vertex set V and edge set E.

(a) If $C \subseteq V$, define what is meant by the **neighbourhood** of C in G. [**2**]

(b) Now suppose G is bipartite, with bipartition (A, B). Define what is meant by a **matching** from A to B. [**3**]

(c) Give a precise statement of Hall's Matching Theorem. [**3**]

(d) Does the graph below have a matching from $\{1, 2, 3, 4, 5\}$ to $\{6, 7, 8, 9, 10\}$? Justify your answer. [**4**]



Question 3. [18 marks] Solve the following recurrence relations, with the given initial conditions.

(a) $a_n = 2a_{n-1} + 3a_{n-2}$, with $a_1 = 1$, and $a_2 = 11$. [**6**]

(b) $b_n = 3b_{n-1} - 3b_{n-2} + b_{n-3}$, with $b_1 = 1$, $b_2 = 2$, and $b_3 = 5$. [**6**]

(c) $c_{n+1} = c_n^3$, with $c_1 = 2$. [**6**]

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Question 4. [24 marks]

(a) Define what it means for a graph to be a **plane** graph. Define what it means for a graph to be a **planar** graph. [**4**]

(b) State Euler's formula for connected plane graphs. [**3**]

(c) Use Euler's formula to show that in a connected bipartite plane graph, the number n of vertices and the number e of edges satisfy

$$e \leqslant 2(n-2).$$

(Hint: Recall that a bipartite graph contains no triangles.) [**6**]

(d) Suppose $m, n \in \mathbb{N}$. Give the definition of the **complete bipartite graph** $K_{m,n}$. [**3**]

(e) Use part (c) to show that $K_{3,3}$ is not planar. [**3**]

(f) For which m, n is $K_{m,n}$ planar? Justify your answer. [**5**]

Question 5. [12 marks]

(a) Define the term **Latin square of order** n, and prove that for every positive integer n there is at least one Latin square of order n. [**6**]

(b) Define what it means for a pair of Latin squares of order n to be **orthogonal**. [**2**]

(c) Let A and B be the following Latin squares of order 4.

$$A = \begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{matrix} \qquad B = \begin{matrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \end{matrix}$$

Find another Latin square of order 4 which is orthogonal to both A and B. [**4**]

Question 6. [14 marks]

(a) State and prove the Principle of Inclusion and Exclusion. [**8**]

(b) Now suppose $n \geqslant k$. Use the Principle of Inclusion and Exclusion to prove that the number of surjective functions from $\{1, \ldots, n\}$ to $\{1, \ldots, k\}$ is

$$\sum_{j=0}^{k-1} (-1)^j \binom{k}{j} (k-j)^n.$$ [**6**]

End of Paper.

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