Queen Mary
University of London

B. Sc. Examination by course unit 2015

MTH6109 Combinatorics

Duration: 2 hours

Date and time: 11 May 2015, 1000h–1200h

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt all questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Candidates should note that the Examination and Assessment Regulations state that possession of unauthorized materials by any candidate who is under examination conditions is an assessment offence. Please check your pockets now for any notes that you may have forgotten that are in your possession. If you have any, then please raise your hand and give them to an invigilator now.

Exam papers must not be removed from the examination room.

Examiner(s): R. A. Wilson

Question 1 The student rugby team has 15 members, and after their match they share two identical crates of beer, each containing 24 bottles.

(a) How many ways are there of distributing the beer to the members of the team? [4]

(b) How many ways are there if everybody has at least one bottle? [4]

(c) How many ways are there if nobody has two (or more) bottles more than anybody else? [4]

(d) How many ways are there if instead there is one crate of beer and one crate of cider? [4]

[Justify your answers. You need not evaluate binomial coefficients in your answers.]

Question 2 A class of mathematics students consists of n men and m women. They wish to choose a committee of k people from the class.

(a) By counting the number of possible committees in two different ways, show that

$$\binom{m+n}{k} = \sum_{r=0}^{k} \binom{m}{r}\binom{n}{k-r}.$$

[4]

(b) The women choose from among themselves a netball team of t players, including a captain. By counting the number of possibilities in two ways, show that

$$t\binom{m}{t} = m\binom{m-1}{t-1}.$$

[4]

(c) Hence, or otherwise, derive a simple formula for

$$\sum_{t=0}^{m} t\binom{m}{t}.$$

[4]

Question 3 (a) Solve the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2}$$

with initial conditions $a_0 = 1$, $a_1 = 2$. [4]

(b) Solve the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2} + 4n - 8$$

with initial conditions $a_0 = 1$, $a_1 = 2$. [8]

Question 4 $S(n,k)$ is defined to be the number of partitions of $\{1,2,\ldots,n\}$ into exactly k (non-empty) parts. Prove the following statements.

(a) $S(n,1) = 1$. [2]

(b) $S(n,n) = 1$. [2]

(c) $S(n,k) = S(n-1,k-1) + kS(n-1,k)$. [4]

(d) $S(n,2) = 2^{n-1} - 1$. [4]

(e) $S(n,n-1) = \binom{n}{2}$. [4]

(f) $S(n,n-2) = n(n-1)(n-2)(3n-5)/24$. [4]

Question 5 (a) Define the terms *bipartite*, *matching* and *neighbourhood*, as used in graph theory. State Hall's Theorem about the existence of matchings in bipartite graphs. [6]

(b) Explain briefly the concept of a 'critical set' and its role in the proof of Hall's Theorem. [4]

(c) Give, with justification, an example of a subgraph of $K_{3,3}$ which has exactly four matchings from one set of three vertices to the other. [4]

Question 6 (a) Prove Euler's equation, that if a connected plane graph has n vertices, p edges, and q faces, then

$$n - p + q = 2.$$

[4]

[You may assume, if you wish, that a tree on n vertices has $n-1$ edges.]

(b) Deduce that, provided $n \geq 3$,

$$p \leq 3n - 6.$$

[6]

(c) Define the *complete graph* K_n, and determine, with justification, for which n the graph K_n is planar, and for which n it is not planar. [4]

Question 7 (a) Define the term *Latin square of order* n, and give an example of a Latin square of order 4. [4]

(b) Explain what it means for two Latin squares of order n to be *orthogonal*, and given an example of a pair of orthogonal Latin squares. [4]

(c) Prove that it is not possible to have more than $n-1$ mutually orthogonal Latin squares of order n. [4]

End of Paper