Queen Mary University of London

Main Examination period 2022 – January – Semester A

MTH6106: Group Theory

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment:

- **You may use books and notes.**
- **You may use calculators and computers, but you must show your working for any calculations you do.**
- **You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.**
- **You must not seek or obtain help from anyone else.**

All work should be **handwritten** and should **include your student number**.

The exam is available for a period of **24 hours**. Upon accessing the exam, you will have **3 hours** in which to complete and submit this assessment.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

Please try to upload your work well before the end of the submission window, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

Examiners: S. Sodin, M. Fayers

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In this paper, we use the following notation: $\mathcal{U}_n$ is the group of residues modulo n which are coprime to n, with the group operation being multiplication modulo n; $\mathcal{D}_{2n}$ is the group with $2n$ elements

$$1, r, r^2, \cdots, r^{n-1}, s, rs, r^2s, \cdots, r^{n-1}s\ ,$$

determined by the relations $r^n = s^2 = 1$, $sr = r^{-1}s$; $\mathcal{S}_n$ is the group of all permutations of $\{1, \cdots, n\}$; $\mathrm{GL}_n(\mathbb{F})$ is the group of invertible $n \times n$ matrices with entries in the field $\mathbb{F}$.

Please justify all your answers!

Question 1 [20 marks]. Let G be a group, and let $a, b \in G$ be such that $ab = b^{2021}a$.

(a) Prove that $ba^{-1} = a^{-1}b^{2021}$.

[**5**]

(b) Let $H = \langle a, b\rangle$, $J = \langle a\rangle$, $K = \langle b\rangle$. One of the following statements is necessarily true. Which one is it?

(i) $J \trianglelefteq H$.

(ii) $K \trianglelefteq H$.

Justify your answer! (You do not need to prove that the other statement is false.)

[**5**]

(c) Let $H = \langle a, b\rangle$, $J = \langle a\rangle$, $K = \langle b\rangle$. One of the following statements is necessarily true. Which one is it?

(i) $\phi : H \to K$, $\phi(b^m a^k) = a^k$, is a homomorphism.

(ii) $\psi : H \to J$, $\psi(b^m a^k) = b^m$, is a homomorphism.

Justify your answer! (You can assume that both maps are well-defined, and you do not need to prove that the other statement is false.)

[**5**]

(d) Give an example of a group G and two elements $a, b \in G$ such that $a, b \neq 1$ and $ab = b^{2021}a$.

[**5**]

Question 2 [20 marks].

(a) Find the order of the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$.

[5]

(b) Find the order of the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_{11})$.

[5]

(c) Find a subgroup of order 5 in $\mathrm{GL}_2(\mathbb{F}_{11})$.

[5]

(d) Let a_0 and $a_1 \in \mathbb{F}_5$ be arbitrary elements, and let $a_2 = a_1 + a_0$, $a_3 = a_2 + a_1$, $a_4 = a_3 + a_2$ and so forth. Prove that $a_m = a_0$ and $a_{m+1} = a_1$, where m is the answer to item (b). *Hint:* use the matrix which we have studied in the previous subquestions to express the vector $\binom{a_{k+1}}{a_k}$ in terms of $\binom{a_k}{a_{k-1}}$.

[5]

Question 3 [20 marks]. Let

$$G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}) \;:\; 2(a+2b) = c+2d \right\} .$$

(a) Prove that G is a group.

[**5**]

(b) Prove that

$$H = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}) \;:\; a+2b = 1 \;, c+2d = 2 \right\}$$

is a normal subgroup of G.

[**5**]

(c) Which two of the following three elements of G lie in the same right coset Hg? Justify your answer.

$$k_1 = \begin{pmatrix} 1 & 3 \\ 10 & 2 \end{pmatrix} \;,\; k_2 = \begin{pmatrix} 3 & 1 \\ 10 & 0 \end{pmatrix} \;,\; k_3 = \begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix} \;.$$

[**5**]

(d) Find a homorphism $\phi : G \to \mathbb{R}^\times$ such that $\ker \phi = H$.

[**5**]

Question 4 [20 marks]. Let $f = (1\,2\,3\,4\,5)(6\,7)(8\,9\,10\,11) \in \mathcal{S}_{11}$.

(a) Is f even or odd? Please justify your answer!

[5]

(b) Find the order of f.

[5]

(c) Compute f^{-1}.

[5]

(d) Write f as a product of 3-cycles.

[5]

Question 5 [20 marks]. The group $\mathrm{SL}_2(\mathbb{F}_3)$ of 2×2 matrices with determinant equal to 1 acts on 2×2 matrices over $\mathbb{F}_3$ by $\pi_g m = gm$.

(a) Find the stabiliser of $m = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

[5]

(b) For the action from (a), find the orbit of the matrix $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$.

[5]

(c) Find the the orbit of the matrix $m = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and check that the answer is consistent with the orbit-stabiliser theorem. You may use without proof that $|\,\mathrm{SL}_2(\mathbb{F}_3)| = 24$.

[5]

(d) How many 3-Sylow subgroups does $\mathrm{SL}_2(\mathbb{F}_3)$ have? You may use that $|\,\mathrm{SL}_2(\mathbb{F}_3)| = 24$.

[5]

End of Paper.