Queen Mary University of London

Main Examination period 2018

MTH6104: Algebraic structures II

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Examiners: Matthew Fayers and Alex Fink

In this paper, we use the following notation.

- $\mathcal{V}_4$ denotes the group $\{1, a, b, c\}$, with group operation given by

$$a^2 = b^2 = c^2 = 1, \qquad ab = ba = c,\ ac = ca = b,\ bc = cb = a.$$

- $\mathcal{U}_n$ is the set of integers between 0 and n which are prime to n, with the group operation being multiplication modulo n.

- $\mathcal{D}_{2n}$ is the group with $2n$ elements

$$1,\ r,\ r^2,\ \dots,\ r^{n-1},\ s,\ rs,\ r^2s,\ \dots,\ r^{n-1}s.$$

  The group operation is determined by the relations $r^n = s^2 = 1$ and $sr = r^{n-1}s$.

- $\mathcal{S}_n$ denotes the group of all permutations of $\{1, \dots, n\}$ (with the group operation being composition). $\mathcal{A}_n$ is the subgroup of $\mathcal{S}_n$ consisting of all even permutations.

Question 1. [20 marks]

(a) Give the definition of a **group**. **[3]**

(b) Let

$$H = \left\{ \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} \;\middle|\; x \in \mathbb{R} \right\}.$$

Prove that H is a group under matrix multiplication. *[You may use standard facts about matrix multiplication.]* **[6]**

Suppose G is a group and $g \in G$.

(c) Give the definition of the **order** of g. **[2]**

(d) Suppose $\operatorname{ord}(g) = 10$ and $m \in \mathbb{N}$. Prove that if $g^m = 1$, then m is divisible by 10. *[You may use standard rules for manipulating powers.]* **[5]**

(e) Give an example of a group G and two elements $g, h \in G$ such that $\operatorname{ord}(g) = \operatorname{ord}(h) = 2$ and $\operatorname{ord}(gh) = 5$. *[You do not need to prove anything.]* **[4]**

Question 2. [16 marks] Suppose G is a group.

(a) Suppose $f, g \in G$. Define what it means to say that f and g are **conjugate** in G. **[2]**

(b) Prove that conjugacy is an equivalence relation on G. **[4]**

(c) Give the definition of a **normal subgroup** of G. *[You do not need to define what a subgroup is.]* **[2]**

(d) Suppose N is a normal subgroup of G. Give the definition of the **quotient group** G/N. *[You do not need to prove anything, but you should say how the group operation on G/N is defined.]* **[2]**

(e) In the case where $G = \mathcal{D}_{12}$ and $N = \{1, r^2, r^4\}$, write down the cosets of N in G and the Cayley table for G/N. *[You may assume that N is a normal subgroup of G.]* **[6]**

Question 3. [18 marks]

(a) Explain how to write an element $f \in \mathcal{S}_n$ in **disjoint cycle notation**. **[3]**

(b) List three advantages of using disjoint cycle notation for permutations. **[3]**

(c) Prove that if $n \geqslant 3$, then $Z(\mathcal{S}_n)$ contains only the identity element. **[5]**

(d) Prove that every element of $\mathcal{A}_n$ can be written as a product of 3-cycles. Write

$$(1\;2\;3\;4)(5\;6\;7\;8\;9)(10\;11\;12\;13)$$

as a product of 3-cycles. **[7]**

Question 4. [15 marks] Suppose G and H are groups.

(a) Give the definition of a **homomorphism** from G to H. **[2]**

(b) Does there exist a homomorphism $\phi : \mathcal{U}_{20} \to \mathcal{U}_{20}$ such that $\phi(3) = 7$ and $\phi(7) = 11$? Justify your answer. **[5]**

Suppose $\phi : G \to H$ is a homomorphism.

(c) Give the definition of the **kernel** and the **image** of ϕ. **[4]**

(d) Write down a homomorphism $\phi : \mathcal{V}_4 \to \mathcal{V}_4$ such that $\mathrm{im}(\phi) = \ker(\phi)$. *[You do not need to prove anything, but you should say where each element of $\mathcal{V}_4$ maps to.]* **[4]**

Question 5. [21 marks]

(a) Suppose G is a group and X is a set. Give the definition of an **action** of G on X. **[3]**

(b) Suppose π is an action of G on X, and $x \in X$. Give the definition of the **orbit** of π containing x. **[2]**

(c) Give two examples of actions of $\mathcal{D}_8$ on itself, one of which is transitive, and the other not transitive. *[You should say clearly how the actions are defined and which one is transitive, but you do not need to prove anything.]* **[5]**

(d) Give a precise statement of the Orbit-Counting Lemma. **[3]**

(e) Suppose we colour the vertices and edges of a square, and we have n colours available. Say that two colourings are equivalent if one can be transformed into the other by a symmetry of the square. How many colourings are there up to equivalence? Justify your answer. **[8]**

Question 6. [10 marks] Suppose G is a group.

(a) Define what it means to say that G is **simple**. **[2]**

(b) Define what is meant by a **composition series** for G. **[3]**

(c) Find a composition series for $\mathcal{D}_{20}$. *[You do not need to prove anything.]* **[5]**

End of Paper.