Main Examination period 2023 – January – Semester A

MTH5123: Differential Equations

Duration: 2 hours

The exam is intended to be completed within **2 hours**. However, you will have a period of **4 hours** to complete the exam and submit your solutions.

You should attempt ALL questions. Marks available are shown next to the questions.

All work should be **handwritten** and should **include your student number**. Only one attempt is allowed – **once you have submitted your work, it is final**.

In completing this assessment:

- You may use books and notes.
- You may use calculators and computers, but you must show your working for any calculations you do.
- You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.
- You must not seek or obtain help from anyone else.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;

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Question 1 [18 marks].

(a) Find the general solution of the the second-order linear ODE

$$2y'' - 5y' - 3y = 0.$$

[**6**]

(b) Find the general solution of the inhomogeneous second-order linear ODE

$$2y'' - 5y' - 3y = 10 \sin x.$$

[**8**]

(c) Find the solution to the Initial Value Problem

$$2y'' - 5y' - 3y = 10 \sin x, \quad y(0) = 4, \;\; y'(0) = 1.$$

[**4**]

Question 2 [26 marks].

(a) Check whether the IVP

$$y' = \frac{x}{y-4}, \quad y(0) = 4.$$

satisfies the hypotheses of the Picard-Lindelöf theorem. [**6**]

(b) Find all the solutions of the IVP defined in (a). Is this result in contradiction with the result obtained in (a)? *Explain your answer.* [**8**]

(c) Determine the smallest $b > 0$ such that the BVP

$$2y'' - 18y = \tanh(x), \quad y(0) = 0, y'(b) = 3,$$

does not have a unique solution. [**12**]

Question 3 [26 marks].

(a) Find the general solution of the motion of a mass attached to the ceiling by a spring in presence of friction, i.e. solve the ODE

$$m\ddot{y} = mg - k(y - l) - \gamma\dot{y}.$$

with $m = 1, k = 3, \gamma = 2$, $g = 10$, $l = 5$, where y indicates the distance of the mass from the ceiling. **[8]**

(b) What is the limit $\lim_{t\to\infty} y(t)$ for the motion of the mass described in (a)? *Describe in words the asymptotic dynamical behaviour of the mass for $t \to \infty$.* **[4]**

(c) Determine whether the differential equation

$$\frac{1}{2}y^2 + y\cos(x) + (yx + \sin(x) - e^y)\, y' = 0$$

is exact. If it is exact, find its general solution in explicit form. **[14]**

Question 4 [30 marks].

Consider a system of two nonlinear first-order ODEs, where x and y are functions of the independent variable t:

$$\dot{x} = 2\tanh(x) - 2x\cos(y) + e^{x+3y} - 1, \quad \dot{y} = 3\cosh(x) - 3e^{xy} + \frac{1}{2}y + \frac{1}{2}\sin(x).$$

(a) Write down in matrix form of the type $\dot{\mathbf{X}} = A\mathbf{X}$ with $\mathbf{X} = (x, y)^\top$ the system obtained by linearisation of the above equations around the point $x = y = 0$. Specify the elements of the matrix A. **[9]**

(b) Find the eigenvalues and eigenvectors of the matrix A obtained in (a). Write down the general solution of the linear system. **[8]**

(c) What type of fixed point is the equilibrium solution $x = y = 0$? Sketch the phase portrait of the linear system. **[7]**

(d) Find the solution of the linear system corresponding to the initial conditions $x(0) = 1$, $y(0) = 0$. Determine the values $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$. **[6]**

End of Paper – An appendix of 1 page follows.

Picard-Lindelöf Theorem.

Let $\mathcal{D}$ be the rectangular domain in the xy plane defined as $\mathcal{D} = (|x-a| \leq A,\ |y-b| \leq B)$ and suppose $f(x,y)$ is a function defined on $\mathcal{D}$ which satisfies the following conditions:

(i) $f(x,y)$ is continuous and therefore bounded in $\mathcal{D}$

(ii) the parameters A and B satisfy $A \leq B/M$ where $M = max_{\mathcal{D}}|f(x,y)|$

(iii) $|\frac{\partial f}{\partial y}|$ is bounded in $\mathcal{D}$.

Then there exists a unique solution on $\mathcal{D}$ to the initial value problem

$$\frac{dy}{dx} = f(x,y), \quad y(a) = b.$$

Exact first-order ODEs:

If the equation

$$P(x,y) + Q(x,y)\,\frac{dy}{dx} = 0$$

is exact, its solution can be found in the form $F(x,y) = Const.$ where

$$P = \frac{\partial F}{\partial x} \quad \text{and} \quad Q = \frac{\partial F}{\partial y}$$

Some derivatives

In the table below, some derivatives are listed

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$1/\cos^2 x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$1/\cosh^2 x$
$\log x$	$\frac{1}{x}$

End of Appendix.