Queen Mary
University of London

Main Examination period 2019

MTH5123: Differential Equations

Duration: 2 hours

Student number										Desk number			

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Write your solutions in the spaces provided in this exam paper. If you need more paper, ask an invigilator for an additional booklet and attach it to this paper at the end of the exam.

You should attempt ALL questions. Marks available are shown next to the questions.

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Examiners: Shabnam Beheshti, Weini Huang

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Question	Mark	Comments
1	/ 20	
2	/ 20	
3	/ 20	
4	/ 20	
5	/ 20	
Total		

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Note for the rest of this exam paper ODE refers to ordinary differential equation.

Question 1. [20 marks] The equation of motion for a falling object of mass m is given by

$$m\frac{dv}{dt} = mg - \gamma v,$$

where $g = 9.8\ m/s^2$ is the acceleration due to gravity, γ is a constant called the drag coefficient and $v = v(t)$ denotes the velocity of the object at time t. Assume $m = 10\ kg$ and $\gamma = 2\ kg/s$.

(a) Find the general solution to this differential equation with the given constants. **[5]**

(b) Now find the specific solution satisfying the initial condition $v(0) = 49$. **[5]**

(c) Draw integral curves in the t-v plane for various initial conditions, including the initial condition $v(0) = 49$. **[5]**

(d) Interpret your graph for this model, explaining briefly the behaviour of your solutions as $t \longrightarrow \infty$ for different initial conditions. **[5]**

Write your solutions here

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Continue-1: solutions to question 1

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Question 2. [20 marks] Consider the following first-order, linear, inhomogeneous initial value problem

$$\begin{cases} y' &= y\tan x + \sin x, \quad -\pi/2 < x < \pi/2 \\ y(0) &= 1 \end{cases} .$$

(a) Find a solution $y = y(x)$ to the initial value problem. **[10]**

(b) Use the Picard-Lindelöf Theorem to justify existence and uniqueness of solutions to the above IVP in an appropriate rectangular domain. **[10]**

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Continue-1: solutions to question 2

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Continue-3: solutions to question 2

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Question 3. **[20 marks]** Consider the differential equation given by

$$x^2 + \frac{f(y)}{xy} + \ln|xy|\,\frac{dy}{dx} = 0.$$

(a) Find all functions $f(y)$ such that the differential equation becomes exact. **[10]**

(b) For the function f which makes the differential equation exact and which further satisfies $f(1) = 1$, solve the equation in implicit form. **[10]**

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Question 4. [20 marks] Consider the following Euler-type equation

$$x^2 \frac{d^2y}{dx^2} - 2y = 0. \qquad (*)$$

(a) Using $x = e^t$ and $z(t) = y(e^t)$, verify that $(*)$ can be rewritten as

$$\ddot{z} - \dot{z} - 2z = 0.$$

Use this equation to find the general solution $y = y(x)$ to $(*)$. **[5]**

(b) Next, consider the Boundary Value Problem (BVP) for the second order inhomogeneous differential equation

$$x^2 \frac{d^2y}{dx^2} - 2y = f(x), \qquad y(1) = 0, \qquad y(2) + 2y'(2) = 0.$$

Formulate the corresponding left-end and right-end initial value problems. You **do not** need to solve the IVPs. **[5]**

(c) Assume that

$$y_L(x) = \frac{1}{3x} - \frac{1}{3}x^2 \qquad \text{and} \qquad y_R(x) = \frac{4}{x}$$

are the solutions to the IVPs in part (b). Write down the Green's function $G(x, s)$ for the BVP in simplified form. **[5]**

(d) Represent the solution to the BVP in terms of the Green's function $G(x, s)$ for the particular choice $f(x) = e^x$. You **do not** need to evaluate the resulting integrals. **[5]**

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Continue-1: solutions to question 4

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Continue-4: solutions to question 4

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Question 5. **[20 marks]** Consider the autonomous dynamical system given by

$$\dot{x} = 4y\,,\ \dot{y} = -x.$$

(a) Rewrite the system in matrix form and find the associated eigenvalues and eigenvectors. **[5]**

(b) Determine the solutions of the corresponding initial value problems for the general initial conditions $x(0) = a, y(0) = b$. **[5]**

(c) Sketch the phase portrait in the (x, y) phase plane and describe the shape of the trajectories in the phase plane. If the initial condition is $(x(0), y(0)) = (3, 4)$, describe the qualitative behaviour of the solution given in the phase portrait. **[5]**

(d) Determine all fixed points of the system and describe the stability of $(x(t), y(t)) = (0, 0)$ as a solution for the linear system. What type of equilibrium point is $(0, 0)$? **[5]**

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End of Paper – An appendix of 1 page follows.

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Appendix

Picard-Lindelöf Theorem. Let $\mathcal{D}$ be the rectangular domain in the xy plane defined as $\mathcal{D} = (|x-a| \leqslant A,\ |y-b| \leqslant B)$ and suppose $f(x,y)$ is a function defined on $\mathcal{D}$ which satisfies the following conditions:

(i) $f(x,y)$ is continuous and therefore bounded in $\mathcal{D}$

(ii) the parameters A and B satisfy $A \leq B/M$ where $M = max_{\mathcal{D}}|f(x,y)|$

(iii) $|\frac{\partial f}{\partial y}|$ is bounded in $\mathcal{D}$.

Then there exists a unique solution on $\mathcal{D}$ to the initial value problem

$$\frac{dy}{dx} = f(x,y), \quad y(a) = b.$$

Green's Function Method. If there exists a unique solution $y(x)$ to the inhomogeneous **boundary value problem**

$$\mathcal{L}(y) = a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x)$$

in an interval $x \in [x_1, x_2]$ with linear homogeneous boundary conditions

$$\alpha y'(x_1) + \beta y(x_1) = 0, \quad \gamma y'(x_2) + \delta y(x_2) = 0,$$

it can be found by the Green's function method:

$$y(x) = \int_{x_1}^{x_2} G(x,s)\, f(s)\, ds, \quad G(x,s) \equiv \begin{cases} A(s)\, y_L(x), & x_1 \leqslant x \leqslant s \\ B(s)\, y_R(x), & s \leqslant x \leqslant x_2 \end{cases}.$$

Here

$$A(s) = \frac{y_R(s)}{a_2(s)W(s)}, \quad B(s) = \frac{y_L(s)}{a_2(s)W(s)}, \quad W(s) \equiv y_L(s)y_R'(s) - y_R(s)y_L'(s)$$

and $y_L(x), y_R(x)$ are solutions to the left/right initial value problems:

$$\mathcal{L}(y) = 0,\ y(x_1) = \alpha,\ y'(x_1) = -\beta; \quad \text{and} \quad \mathcal{L}(y) = 0,\ y(x_2) = \gamma,\ y'(x_2) = -\delta.$$

End of Appendix.

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