Queen Mary
University of London

# B. Sc. Examination by course unit 2015

# MTH 5120: Statistical Modelling I

## Duration: 2 hours

## Date and time: 30.04.2015, 14.30-16.30

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**Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.**

**You should attempt ALL questions. Marks awarded are shown next to the questions.**

**Calculators ARE permitted in this examination. The unauthorised use of material stored in pre-programmable memory constitutes an examination offence. Please state on your answer book the name and type of machine used.**
**Statistical functions provided by the calculator may be used provided that you state clearly where you have used them.**
**The New Cambridge Statistical Tables are provided.**

Complete all rough workings in the answer book and **cross through any work that is not to be assessed**.

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**Exam papers must not be removed from the examination room.**

**Examiner(s): B. Bogacka and L. I. Pettit**

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**Question 1. (25 marks)**

(a) Show that the Least Squares Estimator $\widehat{\beta}$ of the parameter $\beta$ in the no-intercept model $Y_i = \beta x_i + \varepsilon_i$, where the random errors $\varepsilon_i$ are identically, independently normally distributed with zero mean and a constant variance $\sigma^2$, is

$$\widehat{\beta} = \frac{1}{a}\sum_{i=1}^{n} Y_i x_i, \qquad \text{where} \qquad a = \sum_{i=1}^{n} x_i^2.$$

[**8**]

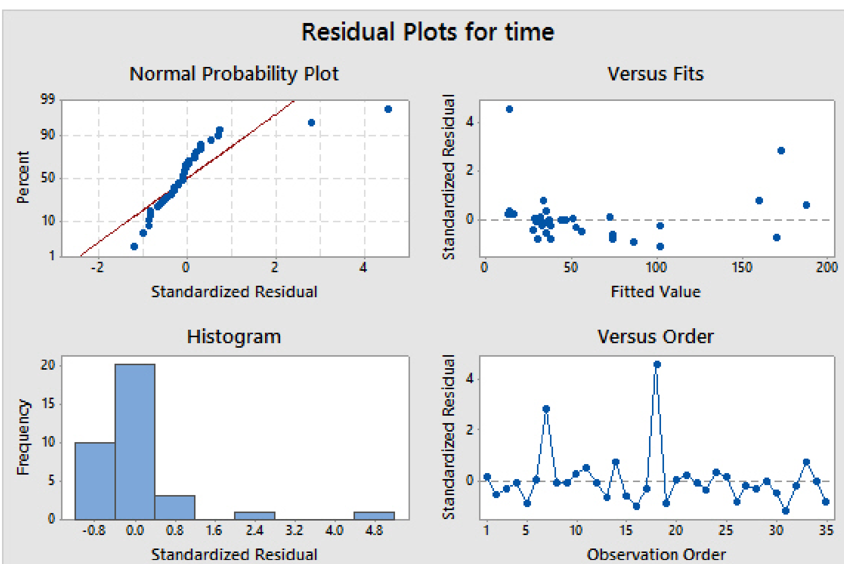
(b) Obtain the distribution of $\widehat{\beta}$ including the mean and the variance of the estimator. [**12**]

(c) Assuming that $\sigma^2$ is known, give a statistic and its distribution for testing the hypothesis $H_0 : \beta = 0$ versus the alternative $H_1 : \beta \neq 0$. [**5**]

**Question 2. (25 marks)**

The *winning times* (Y, [minutes]) in 1984 for 35 Scottish hill races were collected together with the *distance on the map* ($X_1$, [miles]) and the *total height gained during the route* ($X_2$, [feet]). A multiple linear regression analysis was performed and the results are given below.

(a) Briefly comment on the standardized residual plots. [**4**]



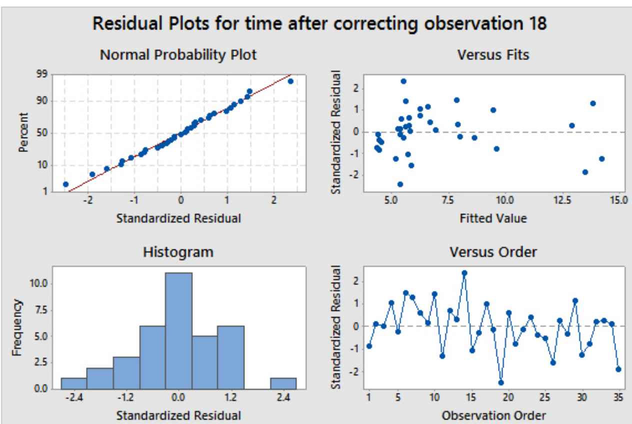

Question 2 continues on the next page.

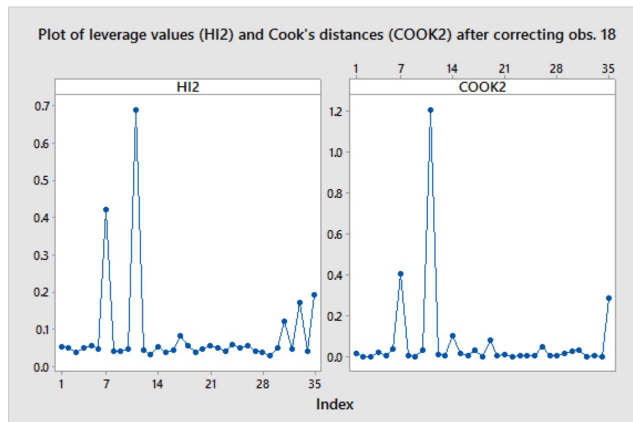
(b) It occurred that observation 18 was wrongly typed in. After correcting the mistake a new regression analysis was performed on the response transformed by power transformation with $\lambda = 0.5$. That is, the assumed model for the independent response variables $Y_i$ was

$$Y_i^{\lambda} = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \varepsilon_i, \qquad \varepsilon_i \sim \mathcal{N}(0, \sigma^2).$$

(i) Briefly comment on the new plots of standardized residuals regarding the assumptions of normality, constant variance and linearity of the model. **[3]**

(ii) Based on the figures shown below, briefly comment on the unusual observations. **[6]**




Analysis of Variance for Transformed Response

| Source | DF | Adj SS | Adj MS | F-Value | P-Value |
|---|---|---|---|---|---|
| Regression | 2 | 261.250 | 130.625 | 553.01 | 0.000 |
| x1 | 1 | 71.003 | 71.003 | 300.60 | 0.000 |
| x2 | 1 | 24.573 | 24.573 | 104.03 | 0.000 |
| Error | 32 | 7.559 | 0.236 | | |
| Lack-of-Fit | 31 | 7.545 | 0.243 | 18.43 | 0.183 |
| Pure Error | 1 | 0.013 | 0.013 | | |
| Total | 34 | 268.809 | | | |

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Regression Equation
time^0.5 = 3.107 + 0.3452 x1 + 0.000693 x2
```

(c) Based on the numerical output shown above do the following:

(i) Test the hypothesis regarding significance of the regression. Give the formula of the test statistic and explain your notation. **[8]**

(ii) Obtain an estimate of the expected record in a Scottish hill race where the distance on the map is 5 miles and the total height gained during the route is 1000 feet. **[2]**

(iii) Interpret in practical terms the meaning of the estimate of $\beta_1$ for a given total height gained during the route. **[2]**

**Question 3. (25 marks)**
Technicians measure the total heat flux as part of a solar thermal energy test. An energy engineer wants to determine how total heat flux ($Y$) is predicted by other variables: insolation ($X_1$), the position of the focal points in the east ($X_2$), south ($X_3$), and north ($X_4$) directions, and the time of day ($X_5$). The best subset regression was performed in Minitab and the results are shown below.

```
Best Subsets Regression: Heat Flux versus Insolation, East, ...

Response is Heat Flux
```

| Vars | R-Sq | R-Sq (adj) | PRESS | R-Sq (pred) | Mallows Cp | S | Insolation | East | South | North | Time of Day |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 72.1 | 71.0 | 4855.9 | 66.9 | 38.5 | 12.328 | | | | X | |
| 1 | 39.4 | 37.1 | 10822.6 | 26.3 | 112.7 | 18.154 | X | | | | |
| 2 | 85.9 | 84.8 | 2736.5 | 81.4 | 9.1 | 8.9321 | | | X | X | |
| 2 | 82.0 | 80.6 | 3786.4 | 74.2 | 17.8 | 10.076 | | | | X | X |
| 3 | 87.4 | 85.9 | 3089.7 | 79.0 | 7.6 | 8.5978 | | X | X | X | |
| 3 | 86.5 | 84.9 | 2725.9 | 81.4 | 9.7 | 8.9110 | X | | X | X | |
| 4 | 89.1 | 87.3 | 2847.2 | 80.6 | 5.8 | 8.1698 | X | X | X | X | |
| 4 | 88.0 | 86.0 | 3045.7 | 79.3 | 8.2 | 8.5550 | X | | X | X | X |
| 5 | 89.9 | 87.7 | 3109.9 | 78.8 | 6.0 | 8.0390 | X | X | X | X | X |

(a) Briefly explain the meaning of all the columns in the above numerical output. Give the formulae for the statistics used. [**8**]

(b) Based on the information in this output suggest the best, from the point of view of prediction, parsimonious subset of the explanatory variables. Briefly justify your choice. [**4**]

Question 3 continues on the next page.

A regression analysis for the full model was performed and a part of the Minitab numerical output is given below.

```
Regression Analysis: Heat Flux versus Insolation, East, South, North, Time of Day

Analysis of Variance
Source         DF   Seq SS  Contribution   Adj SS   Seq MS  F-Value  P-Value
Regression      5  13195.5        89.88%  13195.5  2639.11    40.84    0.000
  Insolation    1   5783.8        39.39%    350.6  5783.78    89.50    0.000
  East          1    811.7         5.53%    270.1   811.72    12.56    0.002
  South         1   1181.5         8.05%    437.2  1181.51    18.28    0.000
  North         1   5303.0        36.12%   4656.6  5303.03    82.06    0.000
  Time of Day   1    115.5         0.79%    115.5   115.50     1.79    0.194
Error          23   1486.4        10.12%   1486.4    64.63
Total          28  14681.9       100.00%

Tests use the sequential sums of squares

Model Summary
      S    R-sq  R-sq(adj)    PRESS  R-sq(pred)
8.03902  89.88%     87.68%  3109.95      78.82%

Coefficients
Term            Coef  SE Coef        95% CI       T-Value  P-Value   VIF
Constant       325.4     96.1  ( 126.6,  524.3)      3.39    0.003
Insolation    0.0675   0.0290  (0.0075, 0.1275)      2.33    0.029  2.32
East            2.55     1.25  ( -0.03,   5.13)      2.04    0.053  1.36
South           3.80     1.46  (  0.78,   6.82)      2.60    0.016  3.18
North         -22.95     2.70  (-28.54, -17.36)     -8.49    0.000  2.61
Time of Day     2.42     1.81  ( -1.32,   6.16)      1.34    0.194  5.37

Regression Equation
Heat Flux = 325.4 +0.0675Insolation +2.55East +3.80South -22.95North
            +2.42TimeofDay
```

(c) State the null and the alternative hypotheses regarding significance of regression. Based on the information shown in the ANOVA table test the null hypothesis. [**4**]

(d) State the null hypotheses for the coefficients of the explanatory variables which are tested in the ANOVA table. [**3**]

(e) State the null hypotheses for the coefficients of the explanatory variables which are tested in the table of Coefficients. [**3**]

(f) Explain how you could improve the model fit. [**3**]

**Question 4. (25 marks)**

(a) For each of the following regression models, indicate whether it is a linear regression model (in the parameters $\beta$). If it is not, state whether it can be linearized by a suitable transformation of the response and write down the transformed model. [**8**]

(i) $Y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 \log_{10} x_{2i} + \beta_3 x_{1i}^2 + \varepsilon_i$

(ii) $Y_i = \varepsilon_i \exp(\beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i})$

(iii) $Y_i = \beta_0 \exp(\beta_1 x_{1i}) + \varepsilon_i$

(iv) $Y_i = \{1 + \exp(\beta_0 + \beta_1 x_{1i} + \varepsilon_i)\}^{-1}$

(b) Consider the linear model $\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, where $\boldsymbol{Y}$ denotes the $n \times 1$ vector of responses, $\boldsymbol{X}$ denotes the $n \times p$ design matrix, $\boldsymbol{\beta}$ is the $p \times 1$ vector of unknown parameters and $\boldsymbol{\varepsilon}$ is the $n \times 1$ vector of uncorrelated random errors with zero mean and constant variance $\sigma^2$.

(i) Show that $\widehat{\boldsymbol{\mu}} = \boldsymbol{X}\widehat{\boldsymbol{\beta}}$, where $\widehat{\boldsymbol{\beta}}$ denotes the least squares estimator of $\boldsymbol{\beta}$, is an unbiased estimator of the expectation of $\boldsymbol{Y}$. [**6**]

(ii) Obtain the variance-covariance matrix of $\widehat{\boldsymbol{\mu}}$. [**6**]

(iii) State the distribution of $\widehat{\boldsymbol{\mu}}$. [**5**]

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**End of Paper.**