Queen Mary
University of London

B. Sc. Examination by course unit 2015

MTH5117: Mathematical Writing

Duration: 2 hours

Date and time: 27th May 2015, 10:00–12:00

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and **cross through any work that is not to be assessed**.

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Exam papers must not be removed from the examination room.

Examiner(s): J. N. Bray

Marks are deducted for incorrect grammar/spelling. In a question, or part of a question, the notation [$\not\in$, n] indicates that the answer should not contain any mathematical symbols whatsoever, apart from numerals. The integer n—when present—prescribes the *approximate* length (in words). In the absence of this notation, mathematical symbols may be used freely.

Question 1 (20 marks). For each of the following mathematical objects provide two levels of description: (i) a coarse description, which only identifies the class to which an object belongs (set, function, etc.) [$\not\in$]; and (ii) a finer description, which describes the object in question as accurately as possible [$\not\in$].

(a) $\{ m \in \mathbb{Z} : m \equiv 1 \pmod 2 \}$. **[5]**

(b) $9^3 + 10^3 = 1^3 + 12^3$. **[5]**

(c) $x^2 - 5x + 6$. **[5]**

(d) $\{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 1 \}$. **[5]**

Question 2 (20 marks). Express each of the following statements with symbols, **using at least one quantifier**.

(a) The function $f : A \to B$ is not surjective. **[4]**

(b) The function $f : \mathbb{R} \to \mathbb{R}$ is odd. **[4]**

(c) The totally ordered set X has no minimal element. **[4]**

(d) The equation $x^2 + 2 = 0$ has (at least) two distinct real solutions. [Note: The fact that this statement is false has no bearing on the question.] **[4]**

(e) For all integers n at least 3, there is no solution in positive integers x, y, z to the equation $x^n + y^n = z^n$. **[4]**

Question 3 (20 marks). In this question, you may combine words and symbols as appropriate.

(a) Explain the concepts of the *image* and the *inverse image* of a set under a function. Provide illustrative examples. **[8]**

(b) Explain the *infinite descent method*. **[6]**

(c) Prove, **using the method of infinite descent**, that for any prime p there are no positive integers m and n such that $n^2 = pm^2$. **[6]**

Question 4 (16 marks). Each of the following definitions has faults. (i) Explain what the faults are; and (ii) write out an appropriate revision.

(a) Let f be the following real function:

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x, y) = \frac{\pm\sqrt{x + y}}{(x + 1)(y + 2)}.$$ **[8]**

(b) Let X be a subset of $\mathbb{R}$, and let $f(X)$ be the number of integers in X. Denoting by $|A|$ the cardinality of the set A, we have:

$$f : \mathbb{R} \to \mathbb{Z}^+, \qquad f(X) = |x \in X \cap x \in \mathbb{Z}|.$$ **[8]**

Question 5 (6 marks). The following lemma has a defective proof.

LEMMA. *There is an invertible 2×2 matrix whose entries lie in $\mathbb{Z}_2$. (Recall that $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ is the set of integers modulo 2.)*

PROOF. For

$$A = \begin{pmatrix} a & b+1 \\ b & a+1 \end{pmatrix}$$

has determinant $a^2 + a - b^2 - b \neq 0$, so done. $\square$

(a) Explain the fault(s); and **[4]**

(b) Give a correct proof. **[2]**

Question 6 (18 marks). Read the text displayed on the next two pages, and then write a report on it, comprising

- a short title **[** $\not\epsilon$ **]**; **[2]**
- two/three concise key points **[** $\not\epsilon$ **]**; **[4]**
- a summary of the document **[** $\not\epsilon$, 175**]**. **[12]**

End of Paper—An appendix of 2 pages follows.

This page and the next contain material for Question 6.

Given two integers d and n, we say that d *divides* n and write $d \mid n$ if there exists an integer q such that $n = dq$. Since for all n we have $n = 1 \cdot n$, it follows that 1 and n divide n, and since $0 = d \cdot 0$, we see that any integer divides 0. Also if d divides n, so does $-d$. For this reason, when dealing with divisibility it is customary to consider the positive divisors only. A *non-trivial divisor* d of n is a divisor that is neither equal to 1 nor to n.

The identity $n = d(n/d)$ shows that if d divides n, so does n/d (because n/d is an integer), and vice-versa. So divisors come in pairs, and therefore the number of divisors of n is even, unless d and n/d happen to coincide, giving $n = d^2$, a square. Thus an integer is a square precisely when it has an odd number of divisors.

A positive integer n greater than 1 is said to be *prime* if it has only two divisors: 1 and n (or, equivalently, if it has no non-trivial divisor). If this is not the case, we say that n is *composite*. Note that 1 is not considered to be either prime (see below) or composite. The following basic result of the arithmetic of the integers is known as the *Fundamental Theorem of Arithmetic*.

Theorem. *Every integer n greater than one can be written as a product of the form*

$$n = p_1^{e_1} \cdot p_2^{e_2} \cdot \cdots \cdot p_k^{e_k}. \tag{1}$$

where the p_i are distinct primes and the exponents e_i are positive integers. The factorisation (1) is unique up to the ordering of the factors.

The primes p_i appearing in (1) are called the *prime divisors* of n. You can see why 1 is not considered prime: if it were, then we could insert the extra factor 1 in the product (1), to obtain a different decomposition of n into primes.

Knowledge of the prime factorisation provides useful information. For instance, when constructing the divisors of n, we find from (1) that there are $e_i + 1$ possible choices for each exponent e_i (from 0 to e_i). So the number $d(n)$ of divisors of n is given by

$$d(n) = (e_1 + 1)(e_2 + 1) \cdots (e_k + 1). \tag{2}$$

If n is a square, then each exponent e_i is even, that is, $e_i + 1$ is odd. Then the above equation shows that $d(n)$ is odd, in agreement with the observation made above.

In fact, we can generalise the above, and for $m \in \mathbb{N}_0$ and n a positive integer, we can define $\sigma_m(n)$ by

$$\sigma_m(n) := \sum_{d>0, d \mid n} d^m.$$

Since $d^0 = 1$ for all d, we observe that $\sigma_0(n) = d(n)$ for all n. (We also write $\sigma(n)$ instead of $\sigma_1(n)$ for the sum of all (positive) divisors of n, including 1 and n.) If p is prime and $e \in \mathbb{N}_0$ then using (1) gives us

$$\sigma_m(p^e) = 1 + p^m + p^{2m} + \cdots + p^{em},$$

and so $\sigma_0(p^e) = e + 1$ (as above) and $\sigma_m(p^e) = \frac{p^{(e+1)m}-1}{p^m-1}$ if $m > 0$. A reasonably straightforward argument generalises the result of (2) to all $m \in \mathbb{N}_0$, namely that

$$\sigma_m(n) = \sigma_m(p_1^{e_1}) \cdot \sigma_m(p_1^{e_1}) \cdot \cdots \cdot \sigma_m(p_k^{e_k}),$$

where n is as in (1). In a similar vein, we also have $\sigma_m(n_1 n_2) = \sigma_m(n_1)\sigma_m(n_2)$ whenever $\gcd(n_1, n_2) = 1$. We thus have a formula for $\sigma_m(n)$, and this is useful for the study of certain phenomena, such as perfect numbers (where n is said to be *perfect* if $\sigma(n) = 2n$).

End of Appendix.