MTH5110 **Introduction to Numerical Computing**

Final exam (marking scheme)

10:00am Monday 18th May, 2015
Duration: 2 hours

Name:
Student ID:

Save the worksheet when you have entered your name and ID (and save the worksheet at regular intervals, say every 5 minutes, to avoid data loss).

When you have finished the exam save the file and send the file from your college email account by email as an attachment to:

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with subject field "<student ID> mth5110 exam".

This is an OPEN BOOK exam, all questions count.

permitted:
- any printed material, e.g. books
- any handwritten notes
- photocopies of any kind
- use of a computer (Maple, google, wikipedia, ...)

prohibited:
- using electronic communication devices (e.g. mobile phones, email - except to submit the exam - , twitter,...)
- sharing material with other students

YOU ARE NOT PERMITTED TO START READING THE SUBSEQUENT PARTS OF THIS QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY AN INVIGILATOR.

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Problem 1

a) **[8 marks]**

Evaluate the expression 114/9 using arithmetic with 2 digits in the mantissa. Explain briefly the outcome.

```
> evalf[2](evalf[2](114)/evalf[2](9));
```

$$12. \tag{1}$$

each evalf (2+2+2 marks) (one not needed)
result mathematically incorrect as 114 has 3 significant digits (2 marks)

b) **[8 marks]**

When computing $\sqrt{\sqrt{2}\,\sqrt{2}-2}$ in Maple we seem to obtain a complex number

```
> sqrt(sqrt(2.0)*sqrt(2.0)-2.0);
```

$$0.00003162277660\, I \tag{2}$$

Explain this finding.

Floating point arithmetics involves truncation errors (4 marks).
sqrt(2.0) givesa result which is slightly too small (4 marks)

c) **[12 marks]**

Trace the following piece of Maple code

```
> for k from 2 to 4 do
     k:=(k-5)*(k-2):
  end do:
```

2 marks for each output, 4 marks for the last one (2+2+2+2+4)

```
> for k from 2to 4 do
     print(k):
     k:=(k-5)*(k-2):
     print(k):
  end do:
  print(k):
```

$$2$$

0
1
4
5 (3)

Problem 2
It can be shown that the integrals

$$I_n = \int_1^{\infty} x^{-3n} e^{-x^3} \, dx$$

obey the recurrence relation

$$I_n = \frac{1}{3e} - \left(n + \frac{2}{3}\right) I_{n+1}$$

a) **[8 marks]**

State with a reason whether the forward iteration is stable or unstable,
and state with a reason whether the backward iteration is stable or unstable.

Forward iteration is stable (2 marks) since 1/(n-2/3)<1 for large n (2 marks).
Backward iteration is unstable (2 marks) since n-2/3 >1 for large n (2 marks).

b) **[8 marks]**
Using a stable iteration scheme compute the value of I_{10} with about three significant digits.

Some seed (2 marks)
Correct iteration formula (2 marks)
Some loop with some limits(2 marks)
Consistent output (2 marks)

```
> In:=1.0;
  for n from 0 to 9 do
    In:=(exp(-1.0)/3.0-In)/(n+2.0/3.0);
  end do;
```

$$In := 1.0$$
$$In := -1.316060279$$
$$In := 0.8632120552$$
$$In := -0.2777195905$$
$$In := 0.1091852921$$
$$In := 0.002880254636$$
$$In := 0.02113168690$$
$$In := 0.01522421902$$

$$In := 0.01400899062$$
$$In := 0.01253278728$$
$$In := 0.01138900273 \quad (4)$$

c) **[8 marks]**

Using error propagation estimate the absolute error of the result you have computed in part b).

>

Estimate of seed error (2 marks)
Error propagation for each step (4 marks)
Final estimate (2 marks)

$$\prod_{n=0}^{9} \frac{1}{\left(n+\frac{2}{3}\right)} < \frac{1}{9!} = \frac{1}{362880}$$

Problem 3

a) **[6 marks]**

State a Newton Raphson map for solving the equation $e^x = x$.

>

General NR map (2 marks)
Rearrange equation for rhs 0 (2 marks)
Correct map (2 marks)

```
> x-(exp(x)-x)/(diff(exp(x)-x,x));
```

$$x - \frac{e^x - x}{e^x - 1} \quad (5)$$

b) **[10 marks]**

The inverse function of the function $f(x) = x\,e^x$ is called the Lambert W-function $W(y)$. Write a procedure which computes the value of $W(y)$ using a Newton Raphson scheme. Your procedure should have a single input, the value of y, and it should return the value of $W(y)$ with at least 3 significant digits.

>

NR code from notes (2 marks)

```
> my_lambert:=proc(y)
    # adjust input                (2marks)
   local xold,xnew,phi,f,tol;
    # tolerance in code           (2 marks)
   tol:=10.0^(-3);
    # adjust seed                 (2 marks)
   xnew:=y;
    # fictitious old value for x to start the loop
   xold:=xnew-tol-1;
```

```
 # NR map for f(x)-y=0                    (2 marks)
f:=x->x*exp(x)-y;
phi:=x->x-f(x)/D(f)(x);
 # loop with termination condition
while abs(xnew-xold)>tol do
    # transcribe x value
  xold:=xnew;
    # NR iteration
  xnew:=phi(xold);
end do;
 # output
return xnew;
end proc;
```

$my\_lambert := \textbf{proc}(y)$ **(6)**

$\quad \textbf{local}\ xold, xnew, \phi, f, tol;$

$\quad tol := 10.0\hat{}(-3);$

$\quad xnew := y;$

$\quad xold := xnew - tol - 1;$

$\quad f := x \to x * \exp(x) - y;$

$\quad \phi := x \to x - f(x) / \mathrm{D}(f)(x);$

$\quad \textbf{while}\ tol < \mathrm{abs}(xnew - xold)\ \textbf{do}\ xold := xnew;\ xnew := \phi(xold)\ \textbf{end do};$

$\quad \textbf{return}\ xnew$

$\textbf{end proc}$

c) **[6 marks]**

Use your procedure to compute the value of $W(2)\,\mathrm{e}^{W(2)}$. What is the absolute error of your result?

```
>
```

evaluation (2 marks)
exact result 2 (2 marks)
absolute error (2 marks)

```
> res:=my_lambert(2.0)*exp(my_lambert(2.0));
```

$$res := 2.000000529 \qquad \textbf{(7)}$$

```
> res-2.0;
```

$$5.29\ 10^{-7} \qquad \textbf{(8)}$$

Problem 4

Consider the definite integral

$$I = \int_0^{\frac{1}{\pi}} \sqrt{x}\cos\left(\frac{1}{x}\right)\,\mathrm{d}x \quad .$$

a) **[10 marks]**

Write a Maple procedure which computes a numerical approximation of the integral using the trapezodial rule. Your procedure should have a single input, the number of subintervals n. Your procedure should return the numerical approximation of I and an estimate of the absolute error of the result (you may use any Maple command to derive the error estimate).

```
>
```

Algorithm from notes (2 marks)
Kernel in code (2 marks)
Correct limits in code (2 marks)
Correct evaluation at endpoint 0 (2 marks)
Absolute error in output (2 marks)

```
> my_int:=proc(n)
      local h,s,a,b,f;
      f:=x->sqrt(x)*cos(1.0/x);
      a:=0.0;
      b:=1.0/evalf(Pi);
       # stepsize
      h:=(b-a)/n;
       # boundary points
      s:=0.0-sqrt(1.0/evalf(Pi));
       # sum over nodes
      s:=s+2*add(f(a+k*h),k=1..n-1);
       # output  and absolute error
      return s*h/2,abs(evalf(int(sqrt(x)*cos(1/x),x=0..1/Pi))-s*h/2)
  ;
  end proc;
```

$my\_int := \mathbf{proc}(n)$ **(9)**

$\quad \mathbf{local}\ h, s, a, b, f;$

$\quad f := x \rightarrow \mathrm{sqrt}(x) * \cos(1.0 / x);$

$\quad a := 0.;$

$\quad b := 1.0 / evalf(\pi);$

$\quad h := (b - a) / n;$

$\quad s := 0. - \mathrm{sqrt}(1.0 / evalf(\pi));$

$\quad s := s + 2 * add(f(a + k * h), k = 1 .. n - 1);$

$\quad \mathbf{return}\ 1/2 * s * h, \mathrm{abs}(evalf(int(\mathrm{sqrt}(x) * \cos(1/x), x = 0 .. 1/\pi)) - 1/2 * s * h)$

$\mathbf{end\ proc}$

b) **[8 marks]**

Produce a plot where you show the absolute error vs. n for $n = 2, 3, 4, ..., 200$.
How does the absolute error depend on n?

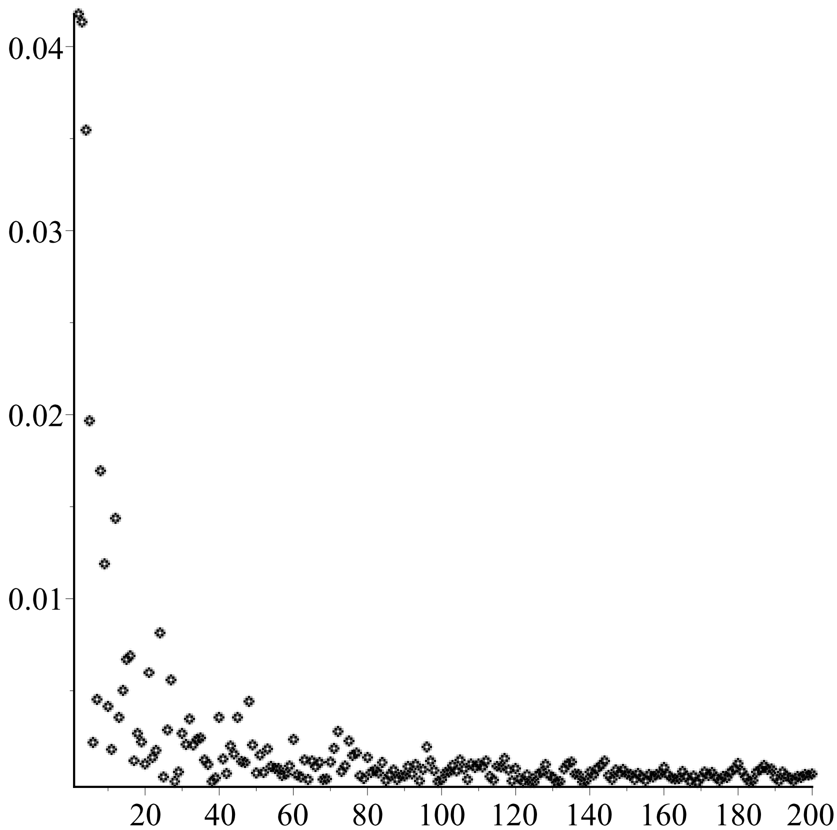
```
>
```

Create list/sequence, e.g. in a loop which contains correct data format, n and error (2 marks)

```
> lst:=NULL:
  for k from 2 to 200 do
     lst:=lst,[k,my_int(k)[2]];
  end do:
```

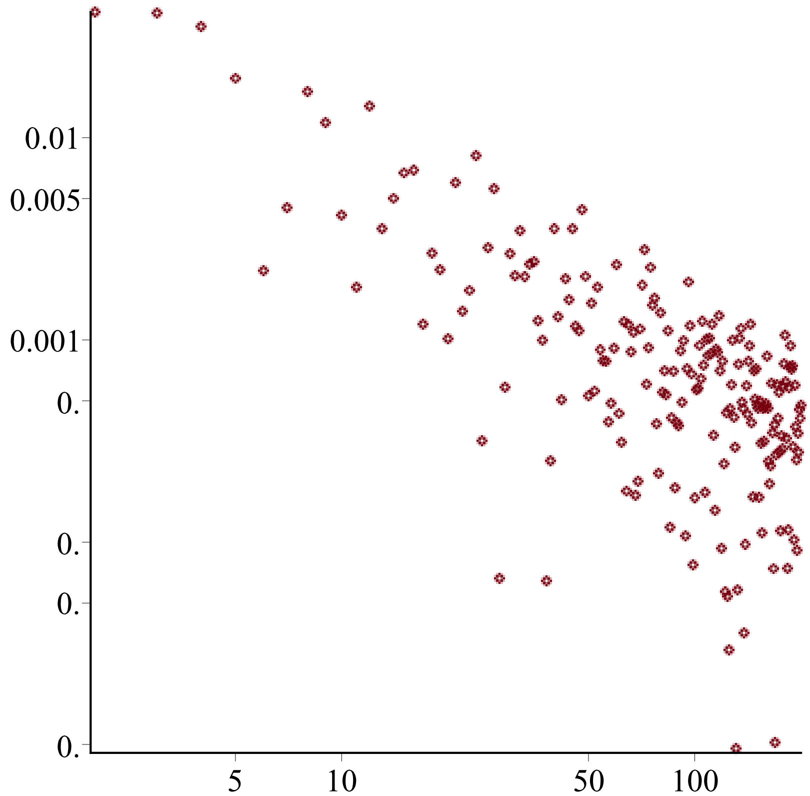
Plot using e.g. listplot (2 marks)

```
> plots[listplot]([lst],style=point);
```



Decays like n^{-1} (4 marks), e.g. logarithmic plot (kernel not smooth and strongly oscillating - full marks for this statement as well).

```
> plots[loglogplot]([lst],style=point);
```



c) **[8 marks]**
Use your procedure to determine the value of I with at least 6 significant digits.

>

At about 200000 nodes are needed, correct input (4 marks)
Run the programm (4 marks)

```
> my_int(200000);
```

$$-0.02316479862,\ 7.0459\ 10^{-7} \quad (10)$$