Queen Mary
University of London

Main Examination period 2020 – May/June – Semester B
Online Alternative Assessments

MTH5105: Differential and Integral Analysis

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment, you may use books, notes, and the internet. You may use calculators and computers, but you should show your working for any calculations you do. You must not seek or obtain help from anyone else.

At the start of your work, please **copy out and sign** the following declaration:

> I declare that my submission is entirely my own, and I have not sought or obtained help from anyone else.

All work should be **handwritten**, and should **include your student number**.

You have **24 hours** in which to complete and submit this assessment. When you have finished your work:

- scan your work, convert it to a **single PDF file** and upload this using the upload tool on the QMplus page for the module;
- e-mail a copy to maths@qmul.ac.uk with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

You are not expected to spend a long time working on this assessment. We expect you to spend about **2 hours** to complete the assessment, plus the time taken to scan and upload your work. Please try to upload your work well before the end of the assessment period, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

Examiners: Huy T. Nguyen, Steve Lester

Question 1 [25 marks].

(a) Let $f : (a, b) \to \mathbb{R}$ be a real valued function. State the definition for f to be **differentiable** at a point $x \in (a, b)$. Geometrically, what does it mean for f to be differentiable at x? **[5]**

(b) Consider the following function, $g : \mathbb{R} \to \mathbb{R}$, given by

$$g(x) = x^2.$$

Using the definition of derivative, compute the derivative of g. **[5]**

(c) Consider the following function, $h : \mathbb{R} \to \mathbb{R}$, $h(x) = |x|$. Show that h is not differentiable at $x = 0$. **[5]**

(d) Let $f : (a, b) \to \mathbb{R}$ be a differentiable function. Prove that f is continuous on (a, b). If f is continuous on (a, b), is it differentiable on (a, b)? (Justify your answer with a proof or counterexample.) **[5]**

(e) Consider a differentiable function $f : \mathbb{R} \to \mathbb{R}$ which satisfies

$$f'(x) = f(x) \quad \forall x \in \mathbb{R}.$$

Using the equation above, show that $f(x) = K \exp(x)$ for some fixed number $K \in \mathbb{R}$. **[5]**

Question 2 [25 marks].

(a) State the definition of a **uniformly continuous function** and explain how it is different from the definition of a continuous function. **[5]**

(b) Prove that $f(x) = x^2$ is uniformly continuous on $[0, 1]$. **[5]**

(c) State the **Mean Value Theorem** and explain its relation to **Rolle's Theorem**. **[5]**

(d) Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Show that if f is non-decreasing (monotone increasing) then $f'(x) \geq 0$. Is the converse statement true? If so, prove the statement; if not, give a counterexample and show that it is a counterexample. **[10]**

Continue to next page

Question 3 [25 marks].

(a) State the **Inverse Function Theorem**. What does this theorem tell us about the relationship between the tangent line to the graph of a function and the tangent line to its inverse? **[5]**

(b) Let $f(x) = \exp(x), x \in \mathbb{R}$. Show that f is invertible and compute the derivative of $f^{-1}(y)$ in terms of y. **[5]**

(c) Let $h : \mathbb{R}\backslash\{-1\} \to \mathbb{R}$ be the function given by

$$h(x) = \frac{1}{1+x}.$$

(i) Using any correct method, find the Taylor series of h about $x = 0$ and find its radius of convergence and interval of convergence. Explain what happens to the function and its Taylor series outside of the radius of convergence. **[7]**

(ii) Compute the anti-derivatives of h. **[2]**

(iii) Hence, using (ii) or otherwise, find the Taylor series and radius of convergence for

$$g(x) = \log(1+x)$$

about $x = 0$. **[6]**

Question 4 [25 marks].

(a) Let $f : [a, b] \to \mathbb{R}$ be a bounded function. State the definition for the **Upper and Lower sum** of f. Explain the relationship of the upper sum and lower sum to the integral of the function. **[5]**

(b) Compute the upper sum $U(g, P_n)$ of $g(x) = x$ for the equidistant partition

$$P_n = \left\{ x_0 = 0, \cdots, x_k = \frac{k}{n}, \cdots, x_n = 1 \right\}. \qquad \textbf{[5]}$$

$\big($*You may use the formula,* $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, *or any other correct method.*$\big)$ Explain what this sum converges to as $n \to \infty$.

(c) State the **Mean Value Theorem for Integrals**. Explain why it is called the mean value theorem. **[5]**

(d) Let $f : [a, b] \to \mathbb{R}$ denote a continuous function and let F, G be antiderivatives of f. Show that F and G differ by a constant. **[5]**

(e) Let $f : [a, b] \to \mathbb{R}$ be a continuous function, show that the function defined by

$$G(x) = \exp\left(\int_a^x f(t) dt \right)$$

is differentiable and find its derivative. **[5]**

End of Paper.