Queen Mary
University of London

Main Examination period 2018

MTH5105: Differential and Integral Analysis

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Examiners: Huy T. Nguyen and Leonard Soicher

Question 1. [25 marks]

(a) Let $f : (a, b) \to \mathbb{R}$ be a real valued function. State the definition for f to be **differentiable** at a point $x \in (a, b)$. **[5]**

(b) Consider the following function, $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = x|x| = \begin{cases} x^2, & x > 0 \\ 0, & x = 0 \\ -x^2, & x < 0 \end{cases}$$

Determine for which $x \in \mathbb{R}$, g is differentiable and, if so, compute its derivative. **[5]**

(c) Show that if f is differentiable at $x \in (a, b)$ then f is continuous at x. **[5]**

(d) Compute the following limits (with full justification) **[5]**

(i) $\lim_{x \to 0} \frac{\sqrt{1+2x} - \sqrt{1-x}}{x}$,

(ii) $\lim_{x \to 0} \frac{\exp(x) - 1 - x}{x^2}$.

(e) Consider a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) = f(x) \quad \forall x \in \mathbb{R}$$

and $f(0) = 1$. Using the property above, show that $f(x)f(-x) = 1$ and that $f(x) \neq 0$ for all $x \in \mathbb{R}$. **[5]**

Question 2. [25 marks]

(a) State the definition of a **uniformly continuous function**. **[5]**

(b) Prove that $f(x) = \frac{x}{x+1}$ is uniformly continuous on $[0, 2]$. **[5]**

(c) State the **Mean Value Theorem**. **[5]**

(d) Show that if $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) that if $f'(x) > 0$ for all $x \in (a, b)$ then f is strictly increasing. **[5]**

(e) Show that if $f : \mathbb{R} \to \mathbb{R}$ is a function such that $|f(x) - f(y)| \leq M|x - y|^2$ for some $M > 0$ and for all $x, y \in \mathbb{R}$ then f is constant. **[5]**

Question 3. [25 marks]

(a) State the **Inverse Function Theorem**. **[5]**

(b) Let $f(x) = \frac{1}{x-1}, x \in (1, \infty)$. Show that f is invertible and if $g(y) = f^{-1}(y)$ is the inverse of f, compute the derivative of $f^{-1}(y)$ in terms of y. **[5]**

(c) Let $h : (-1, 1) \to \mathbb{R}$ be the function given by

$$h(x) = \frac{1}{1+x}.$$

Using any correct method, compute the Taylor series of h about $x = 0$ together with its radius of convergence. **[7]**

(d) For $|x| < 1$, show that

$$\log(1 + x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k.$$ **[8]**

Question 4. [25 marks]

(a) State the **Fundamental Theorem of Calculus**. **[5]**

(b) Let $f_n(x) = \frac{x}{n}, \quad x \in \mathbb{R}$.

(i) Let $f(x) = \lim_{n\to\infty} f_n(x)$. Compute $f(x)$. **[5]**

(ii) Does f_n converge to f uniformly on $[0, 1]$? Justify your answer. **[5]**

(iii) Show that the following limit exists and compute its value,

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx.$$ **[5]**

(c) Assume that $h : [a, b^2] \to \mathbb{R}$ is a continuous function and let $G : [a, b] \to \mathbb{R}$ denote the following function,

$$G(x) = \int_a^{x^2} h(t)dt.$$

Show that G is differentiable and find its derivative.

[5]

End of Paper.