Queen Mary
University of London

Main Examination period 2017

MTH5105
Differential and Integral Analysis

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Examiners: Huy T. Nguyen and Leonard H. Soicher

Question 1. [25 marks]

(a) State the definition of a **uniformly continuous function**. **[5]**

(b) Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[a, 1]$ for any $0 < a < 1$. **[5]**

(c) Let $f_n(x) = x^n, \quad x \in [0, 1]$.

(i) Let $f(x) = \lim_{n\to\infty} f_n(x)$. Compute $f(x)$. **[5]**

(ii) Does f_n converge to f uniformly on $[0, a]$, for $a < 1$? Justify your answer. **[5]**

(iii) Does f_n converge to f uniformly on $[0, 1]$? Justify your answer. **[5]**

Question 2. [25 marks]

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. State the definition for f to be **differentiable at** $a \in \mathbb{R}$. **[5]**

(b) Prove using the definition given in (a) that

$$f(x) = \frac{1}{x^3}$$

is differentiable at $a \in \mathbb{R}\backslash\{0\}$ and compute the derivative of f. **[5]**

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Show that if $|f'(x)| \leq M$ for all $x \in \mathbb{R}$ then

$$|f(x) - f(y)| \leq M|x - y|.$$ **[7]**

(d) Prove, using the Mean Value Theorem, that if $f : (a, b) \to \mathbb{R}$ is differentiable and $f'(x) = 0$ for $x \in (a, b)$ then $f(x) = c$ where c is a constant. **[8]**

Question 3. [25 marks]

(a) Let $f : (a, b) \to \mathbb{R}, a < 0, b > 0$ that is infinitely differentiable at $x = 0$. State the formula for the Taylor series for f about 0. **[5]**

(b) Let $f(x) = \tan(x)$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Show that f is invertible. Let $g(y) = \arctan(y)$ be the inverse of f and compute the derivative of $g(y)$. **[5]**

(c) Let $h : \mathbb{R} \to \mathbb{R}$ be the function given by

$$h(x) = \frac{1}{1 + x^2}.$$

Using any correct method, compute the Taylor series of h about $x = 0$ together with its interval of convergence. **[7]**

(d) Prove for $|y| < 1$ that

$$\arctan(y) = \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{2n + 1}.$$ **[8]**

Question 4. [25 marks] Let a, b be real numbers with $a < b$ and let $f : [a, b] \to \mathbb{R}$ be continuous.

(a) State the Boundedness Principle and the Intermediate Value Theorem. **[8]**

(b) Using part (a), explain why $\int_a^b f(x)dx$ exists and why

$$m = \inf\{f(x) \mid x \in [a, b]\} \quad \text{and} \quad M = \sup\{f(x) \mid x \in [a, b]\}$$

are both finite. **[5]**

(c) With m and M as given in part (b), prove that

$$m \leq \frac{1}{b - a} \int_a^b f(x)dx \leq M.$$ **[5]**

(d) By using the Boundedness Principle, the Intermediate Value Theorem and the previous result, prove that there exists a real number $c \in [a, b]$ such that

$$f(c) = \frac{1}{b - a} \int_a^b f(x)dx.$$ **[7]**

End of Paper.