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Queen Mary University of London

B. Sc. Examination by Course Units

28 May 2014 10 am

MTH5105 Differential and Integral Analysis Duration: 2 hours

YOU ARE NOT PERMITTED TO READ THE CONTENTS OF THIS QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY AN INVIGILATOR.

You should attempt all questions. Marks awarded are shown next to the questions.

CALCULATORS ARE NOT PERMITTED IN THIS EXAMINATION.

COMPLETE ALL ROUGH WORKINGS IN THE ANSWER BOOK AND CROSS THROUGH ANY WORK WHICH IS NOT TO BE ASSESSED.

IMPORTANT NOTE:
THE ACADEMIC REGULATIONS STATE THAT POSSESSION OF UNAUTHORISED MATERIAL AT ANY TIME WHEN A STUDENT IS UNDER EXAMINATION CONDITIONS IS AN ASSESSMENT OFFENCE AND CAN LEAD TO EXPULSION FROM QMUL.

PLEASE CHECK NOW TO ENSURE YOU DO NOT HAVE ANY NOTES, MOBILE PHONES OR UNATHORISED ELECTRONIC DEVICES ON YOUR PERSON. IF YOU HAVE ANY THEN PLEASE RAISE YOUR HAND AND GIVE THEM TO AN INVIGILATOR IMMEDIATELY. PLEASE BE AWARE THAT IF YOU ARE FOUND TO HAVE HIDDEN UNAUTHORISED MATERIAL ELSEWHERE, INCLUDING TOILETS AND CLOAKROOMS IT WILL BE TREATED AS BEING FOUND IN YOUR POSSESSION. UNAUTHORISED MATERIAL FOUND ON YOUR MOBILE PHONE OR OTHER ELECTRONIC DEVICE WILL BE CONSIDERED THE SAME AS BEING IN POSSESSION OF PAPER NOTES. MOBILE PHONES CAUSING A DISRUPTION IS ALSO AN ASSESSMENT OFFENCE.

EXAM PAPERS CANNOT BE REMOVED FROM THE EXAM ROOM.

Examiner: Prof. T. W. Müller

Question 1. a) Let $\mathcal{D} \subseteq \mathbb{R}$ be a set of real numbers, let $f : \mathcal{D} \to \mathbb{R}$ be a real function, and let a be an inner point of $\mathcal{D}$. Define what it means for f to be *differentiable at* a. Give a geometric interpretation of the derivative $f'(a)$, if it exists.

[6 marks]

b) Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \sqrt{x}, & x \geq 0, \\ -\sqrt{-x}, & x \leq 0. \end{cases}$$

Is f differentiable at $a = 0$? Please justify your answer.

[4 marks]

c) Show straight from the definition of differentiability that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 - 3x + 3$ is differentiable everywhere, and that $f'(x) = 2x - 3$ for $x \in \mathbb{R}$.

[15 marks]

Question 2.

a) State Rolle's Theorem. Prove that Rolle's Theorem follows from the fact that a function $f : [a, b] \to \mathbb{R}$, which has a local extremum at $c \in (a, b)$ and is differentiable at c, satisfies $f'(c) = 0$.

[15 marks]

b) State the Mean Value Theorem of differentiation.

[3 marks]

c) Without any computation, show that the roots of the derivative of the function

$$f(x) = x(x-1)(x-2)(x-3)(x-4)$$

are all real. Determine both an upper bound and a lower bound for the set of roots of f'.

[7 marks]

Question 3.

a) Define the *radius of convergence* of a power series $\sum_{n=0}^{\infty} a_n x^n$ with $a_n \in \mathbb{R}$ for all n.

[3 marks]

b) Show the following: if the power series $\sum_{n=0}^{\infty} a_n x^n$ converges for $x = c$, then $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for all $x \in \mathbb{R}$ with $|x| < |c|$.

[9 marks]

c) Prove that the power series $\sum_{n=0}^{\infty} n! x^n$ has radius of convergence 0, i.e., diverges for every real $x \neq 0$.

[6 marks]

d) State Taylor's theorem, and deduce the Mean Value Theorem from it.

[7 marks]

Question 4.

a) State *l'Hôpital's rule*, and use it to compute the limit of

$$\frac{\log(1-x) + x^2}{(1+x)^m - 1}$$

as $x \to 0$. Here m is some positive integer.

[11 marks]

b) Let $f : [a, b] \to \mathbb{R}$ be a bounded function, and let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of the interval $[a, b]$. Define the *upper and lower sum* of f with respect to P.

[4 marks]

c) If P' is a refinement of the partition P in Part (b), show that $L(f, P') \geq L(f, P)$.

[10 marks]

End of Paper