Queen Mary
University of London

Main Examination period 2022 – January – Semester A

MTH5104: Convergence and Continuity

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment:

- **You may use books and notes.**
- **You may use calculators and computers, but you must show your working for any calculations you do.**
- **You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.**
- **You must not seek or obtain help from anyone else.**

All work should be **handwritten** and should **include your student number**.

The exam is available for a period of **24 hours**. Upon accessing the exam, you will have **3 hours** in which to complete and submit this assessment.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

Please try to upload your work well before the end of the submission window, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

Examiners: M. Jerrum, S. Majid

You may assume any standard properties of the functions sin(), cos(), exp() and ln(), including the fact that they are continuous.

Question 1 [20 marks]. In this question, $A, B \subseteq \mathbb{R}$ are non-empty sets of real numbers.

(a) Define what it means for $x \in \mathbb{R}$ to be an **upper bound** for A, and for it to be the **supremum** of A. **[4]**

(b) For each of the specifications below, either give an example of a set A that meets the specification, or state that no such set exists. No justification is required. **[8]**

(i) A countable set that is not bounded above.

(ii) A countable set that is bounded above but has no maximum.

(iii) An uncountable set that is bounded above but has no supremum.

(iv) An uncountable set that has a maximum.

(v) A finite set that has no maximum.

(c) Suppose A has no maximum. Starting with the definition of maximum of a set, prove the following statement:

$$\forall x \in A \ \exists y \in A : y > x.$$ **[4]**

(d) Suppose B has the property that $|x - y| < 1$ for all $x, y \in B$. Prove that B has a supremum. **[4]**

Question 2 [20 marks]. In this question $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty}$, $(u_n)_{n=1}^{\infty}$ and $(v_n)_{n=1}^{\infty}$ are **sequences** of real numbers.

(a) Define (using a quantifier expression) what it means for $(x_n)_{n=1}^{\infty}$ to **converge to** $x \in \mathbb{R}$. **[2]**

(b) Suppose (x_n) converges to $x \in \mathbb{R}$ and that $a, b \in \mathbb{R}$ are real numbers with $a > 0$. Define the sequence $(y_n)_{n=1}^{\infty}$ by $y_n = ax_n + b$ for all $n \in \mathbb{N}$. Prove, **directly from the definition of a convergent sequence**, that (y_n) converges to $ax + b$. **[8]**

(c) Assume now that $(u_n)_{n=1}^{\infty}$ is an increasing sequence. Define the sequence $(v_n)_{n=1}^{\infty}$ by $v_n = \min\{u_n, 1\}$ for all $n \in \mathbb{N}$. Prove that (v_n) converges to a limit. You may appeal to any of the results covered in the course, provided you indicate which you are using. **[6]**

(d) Suppose (u_n) and (v_n) are as in part (c). If (u_n) converges to u and (v_n) to v, what relationship holds between u and v, and why? **[4]**

Question 3 [20 marks]. For each of the **sequences** $(x_n)_{n=1}^{\infty}$ defined in parts (a)–(e), decide if the sequence converges and, if so, to what value. Justify your answers. You may appeal to any of the results covered in the course, provided you indicate which you are using.

(a) $x_n = \dfrac{n+1}{n^2}\big(1+\cos(n\pi/2)\big).$ **[4]**

(b) $x_n = \dfrac{n+1}{n}\big(1+\cos(n\pi/2)\big).$ **[4]**

(c) $x_n = \dfrac{1}{n} + \dfrac{n+1}{n}\big(1+\cos(n\pi/2)\big).$ **[4]**

(d) $x_n = \dfrac{a\sqrt{n}+b}{c\sqrt{n}+d}$, where $a, b, c, d \in \mathbb{R}$ satisfy $c > 0$ and $d \geq 0$. **[4]**

(e) $x_n = \exp\left(\dfrac{n}{n+1}\right).$ **[4]**

Question 4 [20 marks].
Parts (a)–(c) of this question concern the **power series** $\sum_{k=1}^{\infty} x^k/k$. In these parts, you may appeal to any of the results covered in the module, provided you indicate which you are using.

(a) Prove that $\sum_{k=1}^{\infty} x^k/k$ converges absolutely when $|x| < 1$. **[4]**

(b) Prove that $\sum_{k=1}^{\infty} x^k/k$ does not converge when $|x| > 1$. **[6]**
(You may use the fact that $(1+\alpha)^k \geq \alpha k$ for all $k \in \mathbb{N}$ and $\alpha > 0$.)

(c) State whether $\sum_{k=1}^{\infty} x^k/k$ converges when $x = 1$ and $x = -1$ (two cases). No justification is required. **[4]**

(d) For each of the power series (i)–(iii) below, state whether it converges only at $x = 0$, converges for all $x \in \mathbb{R}$, or converges for some non-zero x but not all (in which case, state the radius of convergence). No justification is required.

$$\text{(i)} \sum_{k=1}^{\infty} 2^k k!\, x^k \qquad \text{(ii)} \sum_{k=1}^{\infty} 2^k x^k \qquad \text{(iii)} \sum_{k=1}^{\infty} \frac{2^k}{k!}\, x^k.$$ **[6]**

Question 5 [20 marks].

(a) Define (using a quantifier expression) what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be **continuous** at a point $a \in \mathbb{R}$. **[3]**

(b) Prove **directly from the definition** that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 - x + 2$ is continuous at each point $a \in \mathbb{R}$.

[Given ε, you may like to try δ of the form $\delta = \min\{c\varepsilon, 1\}$ for a suitable constant $c > 0$.] **[8]**

(c) How do you know that the function $g(x) = \exp(x^2 - x + 2)$ is continuous at all points in $\mathbb{R}$? **[3]**

(d) Using the Intermediate Value Theorem (IVT), show that the equation $x^2 - x + 2 = \exp(x)$ has a solution for x in the interval $[0, 1]$. Show explicitly that the conditions of the IVT are satisfied. **[6]**

End of Paper.