Queen Mary
University of London

Main Examination period 2019

MTH5104: Convergence and Continuity

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Examiners: R. Buzano and X. Li

You may assume any standard properties of the sine, cosine and exponential functions including the fact that they are continuous.

Question 1. [20 marks]
Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers.

(a) Define (using quantifier expressions) what it means for $(x_n)_{n=1}^{\infty}$ to **converge** to $x \in \mathbb{R}$. **[3]**

(b) Define (using quantifier expressions) what it means for a real number $x \in \mathbb{R}$ to be an **accumulation point** of $(x_n)_{n=1}^{\infty}$. **[3]**

(c) Define (using quantifier expressions) what it means for $(x_n)_{n=1}^{\infty}$ to be a **Cauchy sequence**. **[3]**

(d) Let $(x_n)_{n=1}^{\infty}$ be the sequence defined by $x_n = (-1)^n$. Prove directly from the definition that 1 is an accumulation point of $(x_n)_{n=1}^{\infty}$. **[5]**

(e) Give an example of a sequence of real numbers which converges to 0. Prove directly from the definition that your example has the desired property. **[6]**

Question 2. [20 marks]

(a) Define **minimum** and **infimum** for a set of real numbers. **[4]**

(b) State the completeness axiom for the set of real numbers. **[3]**

(c) Prove directly from the definitions that $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ has an infimum, but no minimum. **[6]**

(d) For a non-empty set B of real numbers, let $-B = \{-x : x \in B\}$. Suppose B is bounded above. Prove that $-B$ is bounded below, and show $\inf(-B) = -\sup(B)$. **[7]**

Question 3. [20 marks]
You may use any results from the course provided you state clearly which result you are using.

(a) Let $(x_k)_{k=1}^{\infty}$ be a sequence of real numbers. Define what it means for the series $\sum_{k=1}^{\infty} x_k$ to **converge** to $S \in \mathbb{R}$. **[4]**

(b) Which of the following series converge? Justify your answers.

$$\text{(i)} \sum_{k=1}^{\infty} \frac{1}{(2+k)^k}, \qquad \text{(ii)} \sum_{k=1}^{\infty} \frac{\cos(1/k)}{k^2}.$$ **[8]**

(c) Give an example of a series $\sum_{k=1}^{\infty} x_k$ such that $\sum_{k=1}^{\infty} x_k$ converges, but $\sum_{k=1}^{\infty} x_k^2$ does not converge. Justify your answer. **[8]**

Question 4. [20 marks]

(a) Define (using quantifier expressions) what it means to say that a function $f : \mathbb{R} \to \mathbb{R}$ is **continuous** at a point $a \in \mathbb{R}$. **[4]**

(b) Prove directly from the definition that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 + x + 6$ is continuous at $a = 2$. **[8]**

(c) Prove that $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \exp(x) & \text{if } x > 0 \\ \cos(x) & \text{if } x \leq 0 \end{cases}$$

is continuous at all points $a \in \mathbb{R}$. You may use any results from the course provided you state clearly which result you are using. **[8]**

Question 5. [20 marks]

(a) State the Intermediate Value Theorem. **[3]**

Now let $p(x) = x^4 - 5x^2 - 10x - 5$.

(b) Prove that $p(x) = 0$ has at least one solution $x \in [-2, 1]$. **[5]**

(c) Prove that $p(y) = 0$ has at least one solution $y > 1$. **[6]**

(d) Let $p(x)$ be as above and let $q(x) = x^4 - x^3 - 4$. Prove that there exists a real number z with $p(z) = q(z)$. **[6]**

End of Paper.