Queen Mary University of London

Main Examination period 2018

MTH5104: Convergence and Continuity

Duration: 2 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks available are shown next to the questions.

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Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Examiners: R. Buzano and X. Li

You may assume any standard properties of the sine, cosine and exponential functions including the fact that they are continuous.

Question 1. [20 marks]
Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers.

(a) Define (using quantifier expressions) what it means for $(x_n)_{n=1}^{\infty}$ to **converge** to $x \in \mathbb{R}$. **[3]**

(b) Define (using quantifier expressions) what it means for $(x_n)_{n=1}^{\infty}$ to **tend to infinity**. **[3]**

(c) Define (using quantifier expressions) what it means for $(x_n)_{n=1}^{\infty}$ to be a **Cauchy sequence**. **[3]**

(d) Let $(x_n)_{n=1}^{\infty}$ be the sequence defined by $x_n = (-1)^n$. Prove directly from the definition that $(x_n)_{n=1}^{\infty}$ does not converge to any real number. **[6]**

(e) Give an example of a sequence of real numbers which tends to infinity. Show that your example has the desired property. **[5]**

Question 2. [20 marks]

(a) Define **maximum** and **supremum** for a set of real numbers. **[3]**

(b) State the completeness axiom for the set of real numbers. **[3]**

(c) Give an example of a set of real numbers which has a supremum, but no maximum. Explain why your example has the desired properties. **[8]**

(d) Suppose that a non-empty set A of real numbers is bounded below with $\inf(A) > 0$. Let $A^{-1} = \{x^{-1} : x \in A\}$. Prove that A^{-1} is bounded above, and show that $\sup(A^{-1}) = (\inf(A))^{-1}$. **[6]**

Question 3. [20 marks]
You may use any results from the course provided you state clearly which result you are using.

(a) Which of the following series converge? Justify your answers.

$$(i)\ \sum_{k=1}^{\infty} \frac{k^3+k+1}{k^3+2k^2+3}, \qquad (ii)\ \sum_{k=1}^{\infty} \frac{\exp(1/k)}{k^2+3}, \qquad (iii)\ \sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)}.$$ **[12]**

(b) Show that the series $\sum_{k=1}^{\infty} x_k$ given by $x_k = \frac{(-1)^{k+1}}{\sqrt{k}}$ is convergent, but not absolutely convergent. **[8]**

Question 4. [20 marks]

(a) Define (using quantifier expressions) what it means to say that a function $f : \mathbb{R} \to \mathbb{R}$ is **continuous** at a point $a \in \mathbb{R}$. **[3]**

(b) Prove directly from the definition that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x^2 + 5x$ is continuous at $a = 1$. **[7]**

(c) Define (using quantifier expressions) what it means to say that a function $f : \mathbb{R} \to \mathbb{R}$ is **not continuous** at a point $a \in \mathbb{R}$. **[3]**

(d) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is not continuous at 0. Prove that your example has the desired property. **[7]**

Question 5. [20 marks]

(a) State the Intermediate Value Theorem. **[3]**

Now let $p(x) = x^4 - 10x^3 - 5x^2 - 10x - 5$.

(b) Prove that $p(x) = 0$ has at least one solution $x \in [-1, 0]$. **[5]**

(c) Prove that $p(y) = 0$ has at least one solution $y > 0$. **[6]**

(d) Let $x \in [-1, 0]$ and $y > 0$ satisfy $p(x) = 0 = p(y)$ as in (b) and (c). Let $a = \frac{y-x}{2}$. Prove that there exists $z \in [x, \frac{x+y}{2}]$ such that $p(z) = p(z + a)$. **[6]**

End of Paper.