Queen Mary
University of London

MTH5102: Calculus III

Duration: 2 hours

Date and time: Main Examination period 2017

Write your final solutions in the space provided in this exam paper. Additional space for rough work is provided at the end of the paper.

Student number: | | | | | | | | | |

Desk number: | | | |

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiner(s): L. Lacasa

This page is for marking purposes only:

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Section A:

multiple choice & short questions

Section B:

Q6	**Q7**	**Q8**	**Q9**	**Q10**

FINAL MARK:

Section A - multiple choice & short questions

This section is a multiple choice test. Please enter answers by ticking the appropriate box. Use the back of the answer book for rough work. Both questions left unanswered and incorrect answers get zero marks.

Question 1. Consider a sphere of unit radius described implicitly as $f(x,y,z) = 1$, where $f(x,y,z) = x^2 + y^2 + z^2$. A parametric equation of the straight line that is normal to the sphere at the point of coordinates $(1,0,0)$ is **[4]**

☐ $\mathbf{r}(t) = (2t, 0, 0)$
☐ $\mathbf{r}(t) = (0,0,0) + t(2,2,2)$
☐ $\mathbf{r}(t) = (t, t-1, 0)$
☐ None of the above

Question 2. $x^2 + y^2 + 8x - 2y = -16$ is the implicit equation of: **[4]**

☐ a circle centered at $(3,0)$
☐ an ellipse
☐ a circle of radius 1
☐ None of the above

Question 3. Consider the vector field $\mathbf{F} = \mathbf{y}\mathbf{i}$. In spherical coordinates, what is the correct expression for $\mathbf{F}$? **[4]**

☐ $r\sin\theta\sin\phi(\sin\theta\cos\phi\mathbf{e}_r + \cos\theta\cos\phi\mathbf{e}_\theta - \sin\phi\mathbf{e}_\phi)$
☐ $r\sin\theta\sin\phi(\sin\theta\cos\phi\mathbf{e}_r + \sin\phi\mathbf{e}_\theta - \cos\theta\cos\phi\mathbf{e}_\phi)$
☐ $r\sin\theta\sin\phi(\cos\theta\cos\phi\mathbf{e}_r + \sin\theta\cos\phi\mathbf{e}_\theta - \sin\phi\mathbf{e}_\phi)$
☐ $r\sin\theta\sin\phi(\sin\theta\cos\phi\mathbf{e}_r + \cos\phi\mathbf{e}_\theta + \sin\phi\mathbf{e}_\phi)$

Question 4. Consider the vector field $\mathbf{F} = (x^2, xy, xz)$. Compute the divergence and curl of $\mathbf{F}$ and give the solution in the box: **[4]**

Question 5. Let $\mathbf{F}$ and U be a vector field and a scalar field respectively. For each of the following expressions, state whether it is a scalar field (**scalar**), a vector field (**vector**) or an invalid expression (**invalid**): **[4]**

- $U^2(\nabla \times \mathbf{F}) \cdot (\nabla \cdot \mathbf{F})$: ____________________
- $U(\nabla \cdot \mathbf{F})\nabla(\nabla \cdot \mathbf{F})$: ____________________
- $\nabla \times (\nabla U)$: ____________________
- $\nabla \times \nabla(\nabla \cdot \mathbf{F})$: ____________________

Section B

Respond to the following questions only using the space left for that purpose.

Question 6 (15 marks). Let $\mathcal{C}$ be the curve in $\mathbf{R}^3$ whose parametric equation reads $\mathbf{r}(t) = (\cos t, \sin t, 3t)$. Consider the points $A = (0, 0, 0)$, $B = (1, 0, 0)$ and $C = (-1, 0, -3\pi)$.

(a) Make a sketch of this curve, for $0 \leq t \leq 4\pi$. **[2]**

(b) Only two out of these three points belong to the curve. Determine which are these points and compute the arc length of $\mathcal{C}$ between these two points. **[6]**

(c) Consider the vector field $\mathbf{F} = x\mathbf{i} + 4y\mathbf{j} + 6\mathbf{k}$. Calculate the line integral of $\mathbf{F}$ along $\mathcal{C}$ between the two points found in part (b) (the direction should be taken from the point with smaller value of the parameter to the one with larger value). **[7]**

Write your solution to Question #6 below

Additional space for Question 6

Additional space for Question 6

Question 7 (25 marks). Let **F** be a suitably differentiable vector field.

(a) Define what a solenoidal and a conservative vector field are, and then prove the following statement: if **F** is both solenoidal and conservative then its scalar potential ϕ fulfils Laplace's equation. [**6**]

(b) Prove that if **F** is conservative then the circulation of **F** is always null. [**6**]

(c) Prove that if **r**(t) describes a closed curve that is bounding a closed surface, then the line integral of **F** along the whole curve is always null even if **F** is not conservative. [**6**]

(d) Prove that $\mathbf{F} = (2y, 2x - 2y, 6)$ is conservative and compute a scalar potential ϕ for **F**. [**7**]

Write your solution to Question #7 below

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Question 8 (10 marks). (a) State (without proof) the *divergence theorem*. Define precisely all the terms used and any required conditions for the theorem to hold. [**3**]

(b) Apply this theorem to calculate the surface integral of the vector field $\mathbf{F} = zy^2\mathbf{i} + 2y\mathbf{j} - 3x^3\mathbf{k}$ over the cylinder defined by $x^2 + y^2 \leq a^2$, $0 \leq z \leq b$, where a, b are positive constants. Explain in detail why the theorem is applicable in this case. [**7**]

Write your solution to Question #8 below

Additional space for Question 8

Question 9 (15 marks). State the definition of the Fourier series $S(x)$ of a periodic function $f(x)$ of period 2π. [**2**]

Let $f(x)$ be a periodic function of period 2π defined in $(-\pi, \pi]$ by

$$f(x) = \begin{cases} -2 & \text{if } -\pi < x < 0 \\ 2 & \text{if } 0 \leq x \leq \pi. \end{cases}$$

(a) How can you tell easily that $S(x)$ does not have any cosine terms? [**2**]

(b) Calculate the Fourier series $S(x)$ of $f(x)$. [**11**]

Write your solution to Question #9 below

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Question 10 (15 marks). Let $V(x, y)$ be a scalar field that satisfies Laplace's equation inside the square $0 \leq x \leq 2,\ 0 \leq y \leq 2$. The boundary conditions are: $V(x, 2) = \sin 4\pi x$ on the side where $y = 2$, and $V = 0$ on the other three sides.

(a) Make a sketch of the region where $V(x, y)$ is defined, along with the value of the boundary condition on each side. **[3]**

(b) According to the boundary conditions, what type of harmonic function should we use to find the solution to the Dirichlet problem? Justify your answer in detail. **[3]**

(c) Find $V(x, y)$. **[9]**

Write your solution to Question #10 below

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