[Note: The QMplus quiz questions are all new but similar in format/content to those seen during the course; the letters “a” and “b” denote randomly-chosen alternatives for the same question while “r” denotes a question with random numerical parameters. The ordering of answers (but not questions) is shuffled.]

MTH4*07 Semester A Final Assessment 2020/21

1. **Question 1 [8 marks, matching]**

   Match up the following experiments with the cardinalities of their sample spaces.

   (a) You toss a coin. If the coin comes up Tails, you roll a six-sided die. 7
   (b) You roll a six-sided die. If the die shows an odd number, you toss a coin. 9
   (c) You toss a coin three times. 8
   (d) You toss a coin repeatedly until you have seen two Heads or three Tails in total. 10

2. **Question 2 [6 marks, multi-choice]**

   Suppose A and B are events. Which of the following is not necessarily true?

   (a) $\mathbb{P}(A \setminus B^c) = \mathbb{P}(B \setminus A^c)$
   (b) $\mathbb{P}(A \triangle B) = \mathbb{P}(A^c \triangle B^c)$
   (c) $\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c)$
   (d) $\mathbb{P}(A \cap B) = \mathbb{P}(A^c \cup B^c)$ ✓
   (e) None of the other choices

3. **Question 3 [6 marks multi-choice]**

   There are two routes from Arctiville to Bleakopolis and two routes from Bleakopolis to Coldton. Each of the routes is blocked by snow with probability p; the events of them being blocked are mutually independent. What is the probability one can travel from Arctiville to Coldton?

   (a) $(1 - p^2)^2$ ✓
   (b) $(1 - p^4)$

(c) $(1-p)^4$
(d) $2(1-p^2)$
(e) None of the other choices

4. **Question 4a [6 marks, numerical]**

   A marching band consists of 11 pipers and 12 drummers. Three of them are selected to march in order in a scaled-down socially-distanced New Year parade. How many different posssibilities are there if the selected trio must include at least one piper?

   - 9306 ✓

5. **Question 4b [6 marks, numerical]**

   A marching band consists of 11 pipers and 12 drummers. Three of them are selected to march in order in a scaled-down socially-distanced New Year parade. How many different possibilities are there if the selected trio must include at least one drummer?

   - 9636 ✓

6. **Question 5 [6 marks, multi-choice]**

   Under what conditions is an event A independent of itself?

   (a) When $\mathbb{P}(A) = 0$ or $\mathbb{P}(A^c) = 0$ ✓
   (b) When $\mathbb{P}(A) = 0$ and $\mathbb{P}(A^c) = 0$
   (c) When A and A^c are disjoint
   (d) When A and A^c are not disjoint
   (e) None of the other choices

7. **Question 6 [6 marks, multi-choice]**

   Suppose E_1, E_2, and E_3 are mutually independent events with non-zero probabilities. Which of the following is not necessarily true?

   (a) $\mathbb{P}(E_1 \cap E_2 \cap E_3) = \mathbb{P}(E_1)P(E_2|E_1)\mathbb{P}(E_3|E_1 \cap E_2)$
   (b) $\mathbb{P}(E_1^c \cap E_2^c \cap E_3) = \mathbb{P}(E_3) - \mathbb{P}(E_1 \cap E_2)\mathbb{P}(E_3)$ ✓
   (c) $1 - \mathbb{P}(E_1^c)\mathbb{P}(E_2^c \cap E_3^c) = \mathbb{P}(E_1 \cup E_2 \cup E_3)$
   (d) $\mathbb{P}(E_2|E_1)\mathbb{P}(E_3|E_1) = \mathbb{P}(E_2 \cap E_3|E_1)$
   (e) None of the other choices

8. **Question 7r [6 marks, numerical]**

   Let D be the event that a student likes dancing and A be the event that a student likes athletics. Suppose that $\mathbb{P}(D^c|A) = 0.2$, $\mathbb{P}(D|A^c) = 0.7$, and $\mathbb{P}(A^c) = 0.6$. What is the conditional probability that a student likes athletics given they do not like dancing? [Enter the answer correct to two decimal places.]

   - 0.31 ✓

9. **Question 8 [8 marks, multi-choice/numerical]**

   During your revision you stumble across a second-hand copy of the classic textbook "Probability Matters" by the great Professor Damson. [The book is so large that you assume the number of pages is infinite!] You notice that, independently of all other pages, there is a probability of 1/5 that each page contains an example involving a fair die. This means that the number of pages you need to look at to find a die example has a

geometric ✓
binomial
Bernoulli
Poisson

   distribution. If you start reading the book at the beginning, the expected number of pages without a die example before you first encounter one is [4 ✓].

10. **Question 9a [6 marks, multi-choice]**

    A farmer has six geese, each of which lays an egg with probability 2/3 (independently of all the other geese). Suppose that the farmer sells each egg for five gold coins. What is the variance of the number of coins received?

    (a) 100/3 ✓
    (b) 20/3
    (c) 4/3
    (d) 19/3
    (e) None of the other choices

11. **Question 9b [6 marks, multi-choice]**

    There are seven swans in a castle lake. With probability 1/4, each swan attracts a mate from elsewhere. After these new swans have moved in, what is the variance of the number of swans in the lake? [You may assume that the romantic success of each swan is independent of all the others.]

(a) $21/16$ ✓
(b) $21/4$
(c) $133/16$
(d) $49/4$
(e) None of the other choices

12. **Question 10a [8 marks, matching]**

Suppose that X and Y are random variables with $\mathbb{E}(X) = 1$, $\mathbb{E}(Y) = 2$, $\mathrm{Var}(X) = 3$, and $\mathrm{Var}(Y) = 4$. Match up the following quantities.

(a) $\mathbb{E}(X + 3Y)$	7
(b) $\mathbb{E}(X^2 + Y^2)$	12
(c) $\mathrm{Var}(-2Y + 20)$	16
(d) $\mathrm{Var}(X + Y)$	Cannot be determined

13. **Question 10b [8 marks, matching]**

Suppose that X and Y are random variables with $\mathrm{Var}(X) = 1$, $\mathbb{E}(X) = 2$, $\mathrm{Var}(Y) = 3$, and $\mathbb{E}(Y) = 4$. Match up the following quantities.

(a) $\mathbb{E}(X + 3Y)$	14
(b) $\mathbb{E}(X^2 - Y^2)$	-14
(c) $\mathrm{Var}(-2Y + 20)$	12
(d) $\mathrm{Var}(X + Y)$	Cannot be determined

14. **Question 11 [8 marks, multi-choice/multi-choice]**

Suppose V, W, X, and Y are random variables. If V and W are independent then they [must be ✓ / cannot be / may or may not be] uncorrelated. If $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$ then X and Y [must be / cannot be / may or may not be ✓] independent.

15. **Question 12 [6 marks, multi-choice]**

    If X and Y are negatively correlated random variables, which of the following must also hold?

    (a) $-X$ and $-Y$ are positively correlated
    (b) When X takes a negative value, Y takes a positive value.
    (c) $\mathbb{E}(XY) < 0$
    (d) $\text{Var}(2X + Y) < 2\text{Var}(X) + \text{Var}(Y)$
    (e) None of the other choices ✓

16. **Written exam question [20 marks, essay]**

    Dr Harris has recently received the following gifts: one partridge, two turtle doves, and three French hens. Unfortunately, she does not have space for all six birds so she selects two at random to keep.

    (a) What is the probability that she selects two of the same type of bird to keep? [6 marks]

    (b) Let R be the random variable counting the number of partridges kept, and T be the random variable counting the number of turtle doves kept.

        (i) Determine the joint probability mass function of the random variables R and T, and write it in the form of a table with six entries. [4 marks]

        (ii) Explain briefly how you could check your answer to (i). [2 marks]

        (iii) State the name for the distribution of R, and specify any parameters. [2 marks]

    (c) The two birds which are kept are to be fed on pears, from a tree growing by Dr Harris' blackboard. She observes that the number of pears produced each week is a Poisson random variable with the probability of zero pears being 0.1. If a partridge needs to eat two pears per week, while the other birds each need one pear per week, what is the probability that there is *not* enough fruit in a given week? Explain your method carefully and give your final answer to two decimal places. [6 marks]

Your work must be *handwritten* and uploaded as a **single PDF file** with your student ID number on each page.

Notes: (not included in XML)

- (a) [Similar to coursework/lecture example] Treating the problem as unordered sampling without replacement we have $|\mathcal{S}| = \binom{6}{2}$. $\boxed{2}$ There are $\binom{2}{2}$ ways to choose two turtle doves and $\binom{3}{2}$ ways to choose two French hens. $\boxed{2}$ Hence,

$$\mathbb{P}(\text{“two birds the same”}) = \frac{\binom{2}{2} + \binom{3}{2}}{\binom{6}{2}} = \frac{1+3}{15} = \frac{4}{15} \quad \boxed{2}.$$

[Alternatively, use ordered sampling or conditional probabilities.]

(b) [Similar to coursework/lecture example]

(i) Determining the probability mass function via, for example,

$$\mathbb{P}(R = r, T = t) = \frac{\binom{1}{r} \times \binom{2}{t} \times \binom{3}{2-r-t}}{\binom{6}{2}},$$

we have

R \ T	0	1	2
0	3/15	6/15	1/15
1	3/15	2/15	0

. $\boxed{4}$

(ii) The entries in the table must add up to one. The rows give the marginal distribution of R and the columns give the marginal distribution of T; in particular, we must have $\mathbb{P}(R = 0) = \binom{5}{2}/\binom{6}{2}$ and $\mathbb{P}(T = 0) = \binom{4}{2}/\binom{6}{2}$. $\boxed{2}$
[Other answers are possible]

(iii) We have $R \sim \text{Bernoulli}(1/3)$. $\boxed{2}$

(c) [Unseen] Let U be the number of pears per week. We have $U \sim \text{Poisson}(\lambda)$ where $\mathbb{P}(U = 0) = e^{-\lambda} = 0.1$ so $\lambda = -\ln(0.1)$. $\boxed{1}$ Now let F be the event there is not enough fruit. The events “$R = 0$” and “$R = 1$” partition the sample space so, by the law of total probability (Theorem 6.2), we have

$$\begin{aligned}
\mathbb{P}(F) &= \mathbb{P}(F|R=0)\mathbb{P}(R=0) + \mathbb{P}(F|R=1)\mathbb{P}(R=1) \quad \boxed{2} \\
&= \mathbb{P}(U<2) \times \frac{2}{3} + \mathbb{P}(U<3) \times \frac{1}{3} \quad \boxed{2} \\
&= \mathbb{P}(U<2) + \mathbb{P}(U=2) \times \frac{1}{3} \\
&= [\mathbb{P}(U=0) + \mathbb{P}(U=1)] + \mathbb{P}(U=2) \times \frac{1}{3} \\
&= e^{-\lambda} + e^{-\lambda}\lambda + \frac{e^{-\lambda}\lambda^2}{2} \times \frac{1}{3} \quad \boxed{1} \\
&= 0.1 + 0.1(-\ln 0.1) + \frac{0.1(-\ln 0.1)^2}{6} \\
&= 0.42 \quad \text{(to 2 decimal places).} \quad \boxed{1}
\end{aligned}$$

[Alternatively, use the three events "$U < 2$", "$U = 2$" and "$U > 2$" as the partition.]