Queen Mary University of London

B.Sc. EXAMINATION BY COURSE UNIT 2015

MTH4107: INTRODUCTION TO PROBABILITY

Duration: 2 hours

Date and time: 06 May 2015, at 10.00

Examiner: Franco Vivaldi (ext 5488)

Apart from this page, you are not permitted to read the contents of this document until instructed to do so by an invigilator.

You should attempt all questions. Marks awarded are shown next to the questions.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Calculators are NOT permitted in this examination.

Exam papers must NOT be removed from the examination room.

Possession of unauthorised material at any time when under examination conditions is an assessment offence and can lead to expulsion from QMUL. Check now to ensure you do not have any notes, mobile phones, smartwatches or unauthorised electronic devices on your person. If you do, raise your hand and give them to an invigilator immediately. It is also an offence to have any writing of any kind on your person, including on your body. If you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms it shall be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes.

In this paper we employ the following standard notation: Ω is the sample space, ω is an outcome, A, B, C are events, A^c is the complement of A, $\mathbb{P}$ is a probability.

Question 1. [6 marks]

The following expressions are meaningless. Explain what is the fault, as precisely as you can.

i) The product of two events

ii) $A|B$

iii) $\mathbb{P}(A|B^c)^c$

iv) $\Omega \setminus \mathbb{P}(A \setminus \{\omega\})$.

Question 2. [6+5+5+7 marks]

(a) Assuming that the probability of any child being male is 1/2, find the probability that in a family of five children

  i) all children are of the same sex;

  ii) the three oldest are boys and the two youngest are girls;

  iii) three of them are boys and two of them are girls.

(b) I forgot the last digit of my cash card's pin number. I am allowed three attempts, after which the card will be withdrawn. What is the probability that I'll be able to use the card?

(c) Five books, three of which have a yellow cover, are placed at random on a bookshelf. What is the probability that the yellow books will end up side-by-side?

(d) A box contains two fair coins and one unfair coin with two heads. A coin is selected at random from the box, and tossed twice.

  (i) What is the probability that both tosses show heads?

  (ii) If both tosses show heads, what is the probability that it is the unfair coin?

Question 3. [5+5+5 marks]

(a) Prove that $\mathbb{P}(\Omega \setminus A) = 1 - \mathbb{P}(A)$, for any Ω, $\mathbb{P}$ and A, indicating where Kolmogorov's axioms are needed.

(b) Suppose that A_1, A_2, A_3 are events. Show that if $\mathbb{P}(A_1 \cap A_2) > 0$, then $\mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1)\mathbb{P}(A_2|A_1)\mathbb{P}(A_3|A_1 \cap A_2)$.

(c) Let $\Omega = \{1, 2, \ldots, n\}$ and for any subset $A \subset \Omega$ let

$$\mathbb{P}(A) = \begin{cases} 1 & \text{if } 1 \in A \\ 0 & \text{if } 1 \notin A. \end{cases}$$

Show that $\mathbb{P}$ is a probability, namely that $\mathbb{P}$ satisfies Kolmogorov's axioms.

Question 4. [4+4 marks]

(a) Let $\Omega = \{1, 2, \ldots, 20\}$ and let A be the set of even integers in Ω. Assuming that all elements of Ω are equally likely, give an example —with justification— of events B and C such that

(i) A and B are independent;

(ii) A and C are not independent.

(b) Let $\Omega = \{1, 2, 3\}$. Give an example —with justification— of a probability $\mathbb{P}$ for which the events $\{1, 2\}$ and $\{2, 3\}$ are independent.

Question 5. [6+8+6 marks]

(a) Explain what is a *random variable*, and what it means for a random variable to be *discrete*. Let Ω be a set of books; give an example of a discrete random variable on Ω.

(b) A fair die is rolled twice. Let the random variable X be the larger of the two scores.

(i) Compute the probability mass function of X.

(ii) Compute and plot the cumulative distribution function of X.

(c) Let the sample space and random variable X be as in part (b), and let the random variable Y be equal to 1 if the two scores are the same, and equal to 0 otherwise. Compute the joint probability mass function of X and Y.

Question 6. [6+8 marks]

(a) Explain what it means for a random variable to have the *binomial distribution*. Hence state and prove a formula for the expectation of such a random variable.

(b) Two players, Ann and Bob, in turn roll a fair die. Ann rolls first, and the first one to roll a six wins.

i) What are their respective chances of winning?

ii) Let the random variable X be the number of times Bob rolls the die. Compute $\mathbb{P}(X = k)$ for $k \leqslant 3$.

Question 7. [7+7 marks]

(a) Introduce the notion of *variance* of a continuous random variable in approximately half a page. All the concepts that are needed to define the variance (density and cumulative distribution functions, expectation, etc) must also be introduced, in a coherent manner.

(b) Consider the function $\delta : \mathbb{R} \to \mathbb{R}$, given by:

$$\delta(x) = \begin{cases} cx^\alpha & 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

where c and α are real numbers, and $\alpha > -1$.

(i) For given α, determine the value of c such that $\delta = \delta_X$ is the probability density function of some random variable X.

(ii) Compute the expectation and the median of X.

End of Paper