Queen Mary
University of London

Main Examination period 2018

MTH4104: Introduction to Algebra

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Examiners: A.R. Fink, J.N. Bray

Question 1. [10 marks] Find all complex solutions z to the equation

$$z^3 = -3\sqrt{3}$$

and write them in the form $z = a + b\mathrm{i}$ for $a, b \in \mathbb{R}$.

Question 2. [12 marks]

(a) Define what it means for $\mathcal{A} = \{A_1, A_2, \ldots\}$ to be a **partition** of a set X. **[3]**

(b) Let $\mathcal{A}$ be a partition of X. Prove that

$$R = \{\, (x, y) \in X : \text{there exists } i \text{ such that } x \in A_i \text{ and } y \in A_i \,\}$$

is an equivalence relation on X. **[6]**

(c) Write down a partition of $\mathbb{Z}$ into three parts, exactly two of which are infinite. **[3]**

Question 3. [13 marks]

(a) Define the divisibility relation $|$ on the set of natural numbers. **[2]**

(b) A relation R on a set X is said to be **antisymmetric** if the following condition holds: For all elements $a, b \in X$, if $a\,R\,b$ and $b\,R\,a$ both hold then $a = b$. Prove that $|$ is antisymmetric. **[5]**

(c) Define the **least common multiple** of two nonzero natural numbers. **[2]**

(d) Compute the least common multiple of $336 = 2^4 \cdot 3 \cdot 7$ and $180 = 2^2 \cdot 3^2 \cdot 5$. Include an explanation of your method (if you cite facts from lectures or coursework, you need not prove them). **[4]**

Question 4. [24 marks]

(a) Write down the **multiplicative inverse law** for a ring R. [Pay attention to the quantifiers ("for all", "there exists") and other conditions in the law.] **[3]**

(b) Compute the multiplicative inverse of $[23]_{43}$ in $\mathbb{Z}_{43}$. Show your working. **[14]**

(c) Find a multiplicative inverse of the matrix $\begin{bmatrix} [15]_{43} & [14]_{43} \\ [4]_{43} & [11]_{43} \end{bmatrix}$ in $M_2(\mathbb{Z}_{43})$. **[7]**

Question 5. [12 marks]

(a) Give the names of all the axioms that must hold in a **field**. You do not have to write out what the axioms say. **[4]**

(b) Write down the definition of the field $\mathbb{C}$ of complex numbers. You should include a specification of the elements of $\mathbb{C}$ and of its addition and multiplication operations. [You may assume the definition of $\mathbb{R}$ is understood.] **[4]**

(c) Using your definition in part (b), prove that $\mathbb{C}$ satisfies the commutative law for multiplication. [You may assume that $\mathbb{R}$ is a field.] **[4]**

Question 6. [14 marks]

(a) Let R be a ring. Give the definition of a **polynomial in** x **with coefficients in** R. **[2]**

(b) Define the **degree** of a polynomial. **[2]**

(c) Let $f(x)$ and $g(x)$ be nonzero polynomials in $\mathbb{R}[x]$, of degrees m and n, respectively. Prove that $\deg(f(x)\,g(x)) = m + n$. **[5]**

(d) Give a counterexample to the multiplicative inverse law for the ring $\mathbb{R}[x]$ of polynomials in x with real coefficients. Explain why your counterexample works. **[5]**

Question 7. [15 marks]

(a) Define what it means for a set G with a binary operation $*$ to be a **group**. Include statements of any axioms you invoke, not just their names. **[3]**

(b) Let K be the set of integers with the operation $\circ$ defined by

$$x \circ y = x + y + 1.$$

Prove that K with the operation $\circ$ is a group. **[6]**

(c) Let H be a subset of a group $(G, *)$. Define what it means for H to be a **subgroup** of G. **[2]**

(d) Specify a proper subgroup of the additive group $\mathbb{Z}_6$. The Cayley table of $\mathbb{Z}_6$ is provided below. **[4]**

$+$	$[0]_6$	$[1]_6$	$[2]_6$	$[3]_6$	$[4]_6$	$[5]_6$
$[0]_6$	$[0]_6$	$[1]_6$	$[2]_6$	$[3]_6$	$[4]_6$	$[5]_6$
$[1]_6$	$[1]_6$	$[2]_6$	$[3]_6$	$[4]_6$	$[5]_6$	$[0]_6$
$[2]_6$	$[2]_6$	$[3]_6$	$[4]_6$	$[5]_6$	$[0]_6$	$[1]_6$
$[3]_6$	$[3]_6$	$[4]_6$	$[5]_6$	$[0]_6$	$[1]_6$	$[2]_6$
$[4]_6$	$[4]_6$	$[5]_6$	$[0]_6$	$[1]_6$	$[2]_6$	$[3]_6$
$[5]_6$	$[5]_6$	$[0]_6$	$[1]_6$	$[2]_6$	$[3]_6$	$[4]_6$

End of Paper.