Queen Mary
University of London

Main Examination period 2023 – May/June – Semester B

MTH787P / MTH787U: Advanced Derivatives Pricing and Risk Management

Duration: 2 hours

The exam is intended to be completed within **2 hours**. However, you will have a period of **4 hours** to complete the exam and submit your solutions.

You should attempt ALL questions. Marks available are shown next to the questions.

All work should be **handwritten** and should **include your student number**. Only one attempt is allowed – **once you have submitted your work, it is final**.

In completing this assessment:

- You may use books and notes.
- You may use calculators and computers, but you must show your working for any calculations you do.
- You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.
- You must not seek or obtain help from anyone else.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;

Examiners: E. Katirtzoglou, P. Vergel

Show all of your work, briefly justify your answers and clearly indicate any results and formulae you use.
All expressions should be simplified as much as possible.

Question 1 [25 marks].

(a) Consider a stream of payments $\{C_1, \dots, C_n\}$, made at the end of each year, over the next n years. The annual interest rate r is constant and is compounded twice a year. Derive expressions for:

(i) the present value,

(ii) the effective duration, and

(iii) the convexity,

of the cash flows. **[10]**

(b) An investor wants to hedge a liability L by investing into a portfolio of two risky financial instruments Z_1, Z_2 and a risk-free bond. Assume that Z_1 and Z_2 are correlated among each other and are also correlated to the liability L. Assume that $Var[L] = 2$ and,

$$\Sigma_{\mathbf{Z}} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \qquad \Sigma_{L,\mathbf{Z}} = \begin{pmatrix} \gamma \\ -2 \end{pmatrix}.$$

(i) Express the optimal quadratic hedging portfolio (h_0^*, h_1^*, h_2^*) explicitly in terms of the parameter γ and the expected values $\mu_L = \mathbb{E}[L]$, $\mu_1 = \mathbb{E}[Z_1]$, and $\mu_2 = \mathbb{E}[Z_2]$. **[5]**

(ii) Derive an expression for the variance of the hedging error in terms of the parameter γ. **[5]**

(iii) How should Z_1 and L be correlated so that the variance of the hedging error be equal to zero? Calculate possible correlation coefficient(s). **[5]**

Question 2 [25 marks].

(a) You are given an initial capital V_0 and n risky assets, with vector of expected returns $\boldsymbol{\mu}$ and covariance matrix Σ. Consider the efficient frontiers for optimal portfolios that are parametrized by the trade-off parameter c and assume that short-selling is allowed.

(i) In a $\mu - \sigma$ diagram sketch (qualitatively) the efficient frontiers for the optimal portfolios without a risk-free asset and including a risk-free asset. Clearly indicate each frontier. **[2]**

(ii) Explain what different values of the trade-off parameter c represent and the role of c in the $\mu - \sigma$ diagram. **[2]**

(iii) Let $c_1 < c_2 < \mathbf{1}^T\Sigma^{-1}\boldsymbol{\mu}$. Indicate on your sketch two portfolios P_1 and P_2 without risk-free asset that correspond to the trade-off parameters c_1 and c_2, respectively. Explain what happens when the trade-off parameter becomes very small. **[2]**

(iv) Indicate on your sketch a portfolio A that is a combination of the tangent portfolio and the risk free asset. **[2]**

(v) Indicate on your sketch a portfolio B that has the same expected returns as portfolio A. **[2]**

(vi) Based on the quadratic mean-variance approach, which of the two portfolios is preferable, A or B? Explain. **[2]**

(b) The general solution of the portfolio optimization problem including a risk-free asset is

$$\mathbf{w} = \frac{V_0}{c}\Sigma^{-1}(\boldsymbol{\mu} - R_0\mathbf{1}) \tag{1}$$

$$w_0 = V_0 - \mathbf{w}^{\mathrm{T}}\mathbf{1}, \tag{2}$$

where the vector $\boldsymbol{\mu}$ contains the expected returns of the n risky assets and Σ is the covariance matrix of the returns.

(i) Show that the portfolio curve in the $\mu - \sigma$ diagram is a line for all possible values of c. **[6]**

(ii) Where does the line you obtained in the previous part intercept the $\sigma = 0$ axis? Which portfolio corresponds to the point of itersection? **[2]**

(c) Within the context of the quadratic mean-variance approach, give an example of an optimal portfolio where the initial capital is equally distributed among the risky assets. Clearly indicate any assumptions you make and justify your claims. You may use formulae already derived in the lectures. **[5]**

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Question 3 [25 marks].

(a) Give an example and the mathematical properties of a function that could describe:

(i) The utility function of a risk-averse investor. [**3**]

(ii) The utility function of a risk-neutral investor. [**3**]

(iii) The utility function of a risk-loving investor. [**3**]

(b) The *certainty equivalent* c of a random variable X for a given utility function u is defined by the equation $u(c) = \mathbb{E}[u(X)]$. Explain what is the meaning of c

(i) from the perspective of an investor. [**3**]

(ii) from the viewpoint of a mathematician, using Jensen's inequality. [**3**]

(c) A shopping mall owner wishes to manage the risk associated with the possible occurence of a fire in the mall. She considers buying fire insurance for her building for the coming year. The mall owner estimates that the probability of a fire in the following year is p and that the cost of the damage in the case of a fire would be g. Assume that the wealth of the mall owner is given by w and the fire insurance premium by f. Assume that the mall owner is risk-averse and her attitude towards risk is expressed by the utility function $u(x) = a\sqrt{x}$ with $a > 0$ some constant.

(i) Based on the principle of maximal utility derive the premium f^* for which the company is indifferent between buying and not buying fire insurance. [**5**]

(ii) Determine the range of values of the insurance premium that the company is willing to pay. Justify your claim. [**3**]

(iii) How do these values change when a is increased? [**2**]

Question 4 [25 marks].

(a) Describe and discuss, in your own words, the mathematics of the following short-rate interest models: the Cox, Ingersoll and Ross (CIR) model, the Hull-White model and the lognormal model. Provide two distinct examples of their use. [**20**]

(b) In Merton's credit risk model, the risk-neutral evolution of the firm's value V_t is given by $dV_t = rV_t dt + \sigma V_t dW_t$, where r is a constant risk-free interest rate, σ is the volatility of the firm value and dW_t is the Brownian motion with respect to the risk-neutral probability $\mathbb{Q}$. Show that at maturity T, the debt value can be expressed as the difference between a zero coupon bond with face value L and a put option. [**5**]

End of Paper.