Queen Mary
University of London

Main Examination period 2023 – May/June – Semester B

MTH6112: Actuarial Financial Engineering

Duration: 2 hours

The exam is intended to be completed within **2 hours**. However, you will have a period of **4 hours** to complete the exam and submit your solutions.

For actuarial students only: This module also counts towards IFoA exemptions. For your submission to be eligible, **you must submit within the first 3 hours.**

You should attempt ALL questions. Marks available are shown next to the questions.

All work should be **handwritten** and should **include your student number**. Only one attempt is allowed – **once you have submitted your work, it is final**.

In completing this assessment:

- You may use books and notes.
- You may use calculators and computers, but you must show your working for any calculations you do.
- You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.
- You must not seek or obtain help from anyone else.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;

Examiners: L. Fang, F. Parsa

 Continue to next page

Question 1 [30 marks].
Let W_t be a standard Brownian Motion.

(a) The simplest version of the Ornstein-Uhlenbeck process X_t is defined by

$$X_t = \mathrm{e}^{-t} W_t, \quad \text{for some constant } \theta > 0.$$

(i) Does this process have independent increments? [**3**]

(ii) Is X_t a Brownian Motion? [**3**]

(iii) Derive the distribution of the increment $X_t - X_s$ for $t > s$. [**3**]

(iv) Compute $\mu_m = \mathbb{E}[(X_t)^m]$ for all integer $m > 0$. [**3**]

(v) Compute $\mathrm{Cov}(X_t, X_s)$. [**3**]

(b) Consider a Brownian Motion $B_t = \mu t + \sigma W_t$, where W_t is the standard Wiener Process and μ, and σ are the parameters of the Brownian Motion. We also define the related Geometric Brownian S_t by $S_t = \mathrm{e}^{B_t}$.
Are the following processes martingale or not, with respect to the natural filtration, i.e. the one associated with W_t? Please show the detailed calculation to support your answers.

(i) $Z_t = 3W_t$; [**5**]

(ii) $Z_t = W_t^2 - 2t$; [**5**]

(iii) $Z_t = \mathrm{e}^{-\mu t - \frac{\sigma^2 t}{2}} S_t$. [**5**]

Question 2 [20 marks]. A short rate of interest is governed by the Vasicek model, i.e.

$$dr_t = -a(r_t - \mu)dt + \sigma dB_t$$

where B_t is a standard Brownian motion and $a, \mu > 0$ are constants.

You are given that r_t has the following explicit expression:

$$r_t = \mu + (r_0 - \mu)e^{-at} + \sigma \int_0^t e^{a(s-t)} dB_s.$$

(a) Calculate the probability $\mathbb{P}[r_t < 0]$ when $t \to \infty$. Please show the detailed calculation , rather than use the result on relevant slides directly. [**10**]

(b) State what happens to $\mathbb{P}[r_t < 0]$ as $|\sigma| \to 0$. [**5**]

(c) Critically evaluate the Vasicek model. [**5**]

Question 3 [20 marks]. The company F. Bancroft & Sons issued zero-coupon bonds with expiration time of 5 years today, and the total nominal value of £1 million. The total value of the company now stands at £1.2 million. A continuously compounded interest rate is 3% per annum. The total value of the company follows the Geometric Brownian motion with parameters $\mu = 0.3$ and $\sigma = 0.1$.

(a) Give three examples of credit risk models. Which of them are structural model(s)? [**4**]

(b) Under the **Merton model**, calculate the current value of the shareholders' equity. [**8**]

(c) In 2 years time, the company's value drops by 10%. Calculate the probability of F. Bancroft & Sons's default on its obligation to bondholders. [**8**]

Question 4 [30 marks].

The price of a share $S(t)$ evolves according to a Geometric Brownian Motion with parameters $S,\ \mu,\ \sigma$, i.e. $S(t) = Se^{\mu t+\sigma W(t)}$. The continuously compounded interest rate is r.

An exotic derivative on this share has the payoff function

$$R(T) = \frac{1}{T}\int_0^T S(t)(S(T) - c)dt.$$

Where c is a constant. The payoff time is T.

(a) Prove that $\mathbb{E}\left(e^{aW(t)+bW(t+s)}\right) = e^{\frac{(a+b)^2}{2}t+\frac{b^2}{2}s}$, where $t > 0$ and $s > 0$. **[10]**

(b) Use the result obtained in (a), calculate the no-arbitrage price of this exotic derivative. **[10]**

(c) Consider another exotic call option on the same share with expiration time t.

Its strike price K depends on $S(s)$ and $W(s)$, where $s < t$, i.e.

$$K = e^{-\sigma W(s)} \times (S(s))^2.$$

Denote by $\tilde{C}$ the no-arbitrage price of this option.

Denote by $C(S, T, K, \sigma, r)$ the Black-Scholes price of the standard European call option.

Using the properties of the Wiener process, derive the expression for the price $\tilde{C}$ in terms of the expectation of the risk-neutral process $\tilde{S}(t)$ and $\tilde{\mu}$, and show $\tilde{C} = C(S', T', K', \sigma', r')$. (Please write down the explicit expression of S', T', K', σ', r'). **[10]**

End of Paper.