Queen Mary University of London

Main Examination period 2020 – January – Semester A

MTH6105: Algorithmic Graph Theory

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Examiners: F. Fischer, J. Ward

In this paper, $V(G)$ denotes the set of vertices of a graph or digraph G, $E(G)$ the set of edges of a graph G, and $A(G)$ the set of arcs of a digraph G.

Question 1 [27 marks].

(a) Explain what it means for a graph G to be a **tree**. [**2**]

(b) State the tree induction lemma. [**3**]

(c) Using the tree induction lemma or otherwise, prove that $|E(T)| = |V(T)| - 1$ for every tree T. [**5**]

(d) Explain what it means for a tree T to be a **spanning tree** of a graph G, and what it means for a tree T to be a **minimum spanning tree** of a network (G, w). [**3**]

Now consider the network (G, w) such that $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, $E(G) = \{v_1v_2, v_1v_3, v_2v_3, v_2v_4, v_2v_5, v_3v_5, v_3v_6, v_4v_5, v_5v_6\}$ and

$$\begin{array}{lll} w(v_1v_2) = 1, & w(v_1v_3) = 4, & w(v_2v_3) = 4 \\ w(v_2v_4) = 2, & w(v_2v_5) = 3, & w(v_3v_5) = 4 \\ w(v_3v_6) = 1, & w(v_4v_5) = 1, & w(v_5v_6) = 3. \end{array}$$

(e) Use Kruskal's algorithm to determine a minimum spanning tree of this network. Explain clearly what the algorithm is doing and draw the minimum spanning tree. [**10**]

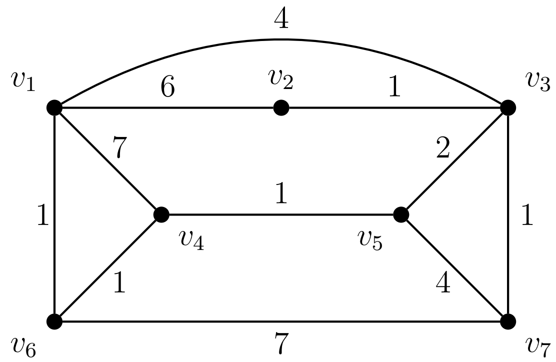
(f) Show that the spanning tree found in the previous part of the question is the unique minimum spanning tree of the network (G, w). You may use any result from the lecture notes without proof. [**4**]

Question 2 [23 marks].

(a) Define the concept of an s–t-**path** in a graph G. [**3**]

(b) Explain what it means for a path to be a **shortest** s–t-**path** in a network (G, w). [**2**]

Recall that Dijkstra's algorithm, when applied to a network (G, w) starting from $s \in V(G)$, constructs a spanning tree T of the connected component of G containing s. Consider the network (G, w) given by the following drawing, where each edge $e \in E(G)$ is labeled by its weight $w(e)$.

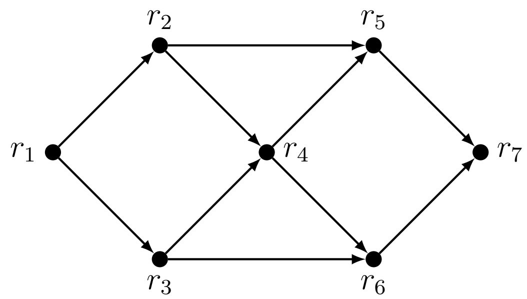


(c) Apply Dijkstra's algorithm to the network starting from vertex v_1. Give $V(T)$ and $E(T)$ after each iteration of the algorithm. [**10**]

(d) Give a shortest v_1–v_7-path in (G, w). What is the length of this path? [**4**]

(e) Give an asymptotic upper bound on the running time of Dijkstra's algorithm when it is applied to an arbitrary network (G, w). Justify your answer and explain why the algorithm is a polynomial-time algorithm. [**4**]

Question 3 [27 marks]. Assume that seven roundabouts r_1 to r_7 are connected by ten one-way streets as shown in the following illustration.



The following table lists for each street the maximum number of cars that can travel along the street per second.

street	r_1r_2	r_1r_3	r_2r_4	r_2r_5	r_3r_4	r_3r_6	r_4r_5	r_4r_6	r_5r_7	r_6r_7
maximum number of cars per second	6	5	3	2	4	3	3	3	4	7

(a) Let (D, c) be the directed network in which each arc $e \in A(D)$ represents a street and the capacity $c(e)$ is equal to the maximum number of cars that can travel along the street. Draw this network. **[2]**

Assume now that the average number of cars per second currently traveling along each street is being measured as follows.

street	r_1r_2	r_1r_3	r_2r_4	r_2r_5	r_3r_4	r_3r_6	r_4r_5	r_4r_6	r_5r_7	r_6r_7
current number of cars per second	4	5	3	1	3	2	3	3	4	5

(b) Let $f : A(D) \to \mathbb{R}$ be the function that maps each arc of the network to the number of cars that currently travel along the street represented by the arc. Show that f is an r_1–r_7-flow for the network (D, c) and give its size. **[5]**

(c) Draw the residual network for network (D, c) and flow f. **[6]**

(d) Using the residual network or otherwise, find a maximum r_1–r_7-flow for (D, c). Explain your reasoning. **[4]**

(e) Argue that the flow you have found is indeed a maximum r_1–r_7-flow. In doing so, you may use any result from the lecture notes without proof. **[4]**

(f) Assume that due to an accident, the number of cars that can travel along the street from r_2 to r_4 is temporarily reduced from 3 to 1. What does this mean for the maximum amount of traffic that can flow from r_1 to r_7? Explain your reasoning. **[3]**

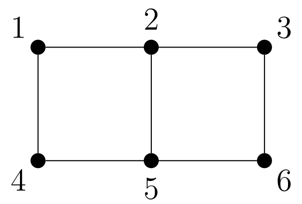
(g) Assume instead that road improvement works could increase the capacity of the street from r_2 to r_4. Would this increase the maximum amount of traffic that can flow from r_1 to r_7? Explain your reasoning. **[3]**

Question 4 [23 marks].

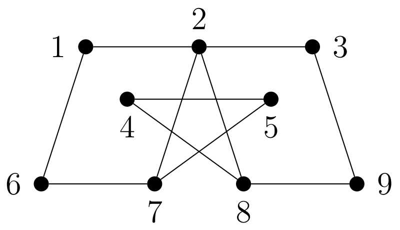
(a) Explain what it means for a graph G to be **bipartite**. **[2]**

(b) For each of the following graphs, state whether the graph is bipartite or not. Justify your answer. In your justification, you may use any result from the lecture notes without proof. **[4]**

(i)



(ii)



(c) Prove that every tree is a bipartite graph. You may use any result from the lecture notes without proof. **[3]**

(d) Define the concept of a **matching** M of a graph G. **[2]**

(e) State Hall's theorem concerning the existence of a matching that saturates one side of a bipartite graph. **[4]**

(f) You are scheduling a set of job interviews, each lasting one hour. There are six candidates – Alice, Bob, Cynthia, Dmitiri, Erica, and Faiz – and six time slots starting at 1pm, 2pm, 3pm, 4pm, 5pm, and 6pm. Not all candidates are available in every time slot, and the time slots where each candidate is available are marked with X in the following table.

	1pm	2pm	3pm	4pm	5pm	6pm
Alice	X	X		X		
Bob		X	X			X
Cynthia	X	X				
Dmitiri		X		X		
Erica	X			X		
Faiz			X		X	X

Your objective is to assign each candidate an interview time such that
(i) candidates are only interviewed at times when they are available and (ii) no two candidates are interviewed at the same time. Find such an assignment, or explain why no such assignment exists. **[8]**

End of Paper.