Queen Mary
University of London

Main Examination period 2022 – May/June – Semester B

MTH5113: Introduction to Differential Geometry

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment:

- **You may use books and notes.**
- **You may use calculators and computers, but you must show your working for any calculations you do.**
- **You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.**
- **You must not seek or obtain help from anyone else.**

All work should be **handwritten** and should **include your student number**.

The exam is available for a period of **24 hours**. Upon accessing the exam, you will have **3 hours** in which to complete and submit this assessment.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

Please try to upload your work well before the end of the submission window, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

Examiners: A. Shao, S. Beheshti

Question 1 [22 marks]. Consider the curve

$$C = \{(x, y, z) \in \mathbb{R}^3 \mid z = x\cos x, y = x\sin x, 0 < x < 6\pi\},$$

and consider the following parametrisation of C:

$$\gamma : (0, 6\pi) \to C, \qquad \gamma(t) = (t, t\sin t, t\cos t).$$

(a) Sketch the image of γ. [**6**]

(b) Compute the tangent line to C at the point $\left(\frac{3\pi}{2}, -\frac{3\pi}{2}, 0\right)$. In addition, **draw and label this on your sketch from part (a).** [**8**]

(c) Assume that C is also given the "leftward" orientation, that is, the orientation of decreasing x-value. Compute the curve integral

$$\int_C \mathbf{F} \cdot \mathbf{ds},$$

where $\mathbf{F}$ is the vector field on $\mathbb{R}^3$ given by

$$\mathbf{F}(x, y, z) = (y^2 + z^2, 0, 0)_{(x,y,z)}.$$ [**8**]

Question 2 [22 marks]. Consider the surface

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + 4y^2 + 4z^2 = 4\},$$

and consider the following parametrisation of S:

$$\sigma : (0, \pi) \times (0, \pi) \to S, \qquad \sigma(u, v) = (2 \cos u \sin v,\ \sin u \sin v,\ \cos v).$$

(a) Sketch the image of σ. Moreover, on your sketch, indicate (i) one path obtained by holding v constant and varying u, and (ii) one path obtained by holding u constant and varying v. **[9]**

(b) Find the unit normals to S at the point $(0, 1, 0)$. **[7]**

(c) Consider the surface integral

$$\iint_S F \cdot dA,$$

where F is the real-valued function

$$F : \mathbb{R}^3 \to \mathbb{R}, \qquad F(x, y, z) = 4 - x^2 - 4y^2 - 4z^2.$$

Find the value of this surface integral. **[6]**

Question 3 [25 marks].

(a) Find the minimum and maximum values of the function

$$a : \mathbb{R}^2 \to \mathbb{R}, \qquad a(x, y) = x^2 y,$$

subject to the constraint

$$x^2 + y^6 = 1.$$

Also, at which points are these minimum and maximum values achieved? [**19**]

(b) Which of the following surfaces are bounded?

$$S_1 = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 1\},$$
$$S_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + 2z^2 = 4\},$$
$$S_3 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - 2z^2 = 4\}.$$

[**6**]

Question 4 [31 marks].

(a) Give a parametrisation of the surface

$$Q = \{(x, y, z) \in \mathbb{R}^3 \mid x^4 + 2y^4 - z = 2\}$$

whose image contains the point $(0, -1, 0)$. **[6]**

(b) Consider the curve

$$L = \{(x, y) \in \mathbb{R}^2 \mid y = 2x\},$$

and consider the following three parametric curves:

$$\begin{aligned} \gamma_1 &: \mathbb{R} \to \mathbb{R}^2, & \gamma_1(t) &= (t^4, 2t^4), \\ \gamma_2 &: \mathbb{R} \to \mathbb{R}^2, & \gamma_2(t) &= (-2t, -4t), \\ \gamma_3 &: \mathbb{R} \to \mathbb{R}^2, & \gamma_3(t) &= (2t, t). \end{aligned}$$

(i) Only one of γ_1, γ_2, γ_3 is a parametrisation of L. Which one is it?

(ii) Why do the other two parametric curves fail to be parametrisations of L? **[6]**

(c) Let C_r denote the circle of radius r about the origin,

$$C_r = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = r^2\}.$$

Apply **Green's theorem** to compute the area of the region bounded by C_r using a curve integral over C_r. **[7]**

(d) Let V be the solid half-ball,

$$V = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 < 1,\ x < 0\}.$$

Describe the boundary of this region V, as one or more surfaces in $\mathbb{R}^3$. (You should express these curves as sets, or you can draw them for partial credit.) **[6]**

(e) Is the following parametric curve regular:

$$\rho : (0, 2\pi) \to \mathbb{R}^2, \qquad \rho(t) = (\cos(3t)\cos t, \cos(2t))?$$

Justify your answer. **[6]**

End of Paper.