Queen Mary
University of London

Main Examination period 2020 – May/June – Semester B
Online Alternative Assessments

MTH5113: Introduction to Differential Geometry

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment, you may use books, notes, and the internet. You may use calculators and computers, but you should show your working for any calculations you do. You must not seek or obtain help from anyone else.

At the start of your work, please **copy out and sign** the following declaration:

> I declare that my submission is entirely my own, and I have not sought or obtained help from anyone else.

All work should be **handwritten**, and should **include your student number**.

You have **24 hours** in which to complete and submit this assessment. When you have finished your work:

- scan your work, convert it to a **single PDF file** and upload this using the upload tool on the QMplus page for the module;
- e-mail a copy to maths@qmul.ac.uk with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

You are not expected to spend a long time working on this assessment. We expect you to spend about **2 hours** to complete the assessment, plus the time taken to scan and upload your work. Please try to upload your work well before the end of the assessment period, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

Examiners: A. Shao, S. Majid

There is a compendium of definitions and formulae in the appendix, which you are free to use without comment.

Question 1 [20 marks]. Consider the curve

$$C = \{(t\cos t, t\sin t, -t) \in \mathbb{R}^3 \mid 0 < t < 4\pi\},$$

and consider the following parametrisation of C:

$$\gamma : (0, 4\pi) \to C, \qquad \gamma(t) = (t\cos t, t\sin t, -t).$$

(a) Sketch the image of γ. **[6]**

(b) Find the tangent line to C at the point $(-3\pi, 0, -3\pi)$. **Draw this on your sketch from part (a).** **[7]**

(c) Compute the curve integral

$$\int_C \mathbf{F} \cdot \mathbf{ds},$$

where C is given the **upward** orientation (in the direction of **increasing** ***z*-value**), and where $\mathbf{F}$ is the vector field given by

$$\mathbf{F}(x, y, z) = (y, -x, z^2)_{(x,y,z)}, \qquad (x, y, z) \in \mathbb{R}^3.$$ **[7]**

Question 2 [16 marks].

(a) Is the following parametric curve regular:

$$\alpha : (0, \infty) \to \mathbb{R}^2, \qquad \alpha(t) = (e^t, \ln t)?$$

Justify your answer. **[5]**

(b) Is the following set a curve:

$$L = \{(x, y) \in \mathbb{R}^2 \mid (x - y)^2 = 0\}?$$

Briefly justify your answer. **[5]**

(c) Find the unit normals to the curve

$$P = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^3 = 1\}$$

at the point $(1, 0)$. **[6]**

Question 3 [21 marks]. Consider the surface

$$S = \{(u, u, u^2 v) \in \mathbb{R}^3 \mid 0 < u < 1,\ 0 < v < 1\},$$

and consider the following parametrisation of S:

$$\sigma : (0,1) \times (0,1) \to \mathbb{R}^3, \qquad \sigma(u, v) = (u,\ u,\ u^2 v).$$

(a) Sketch the image of σ. On your sketch, draw (i) one path obtained by holding v constant and varying u, and (ii) one path obtained by holding u constant and varying v. [**8**]

(b) Find the tangent plane to S at the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{8})$. [**6**]

(c) Compute the surface integral

$$\iint_S H \, dA,$$

where H is the function

$$H : \mathbb{R}^3 \to \mathbb{R}, \qquad H(x, y, z) = xyz.$$ [**7**]

Question 4 [16 marks].

(a) Show that the following parametric surface is regular:

$$\rho : \mathbb{R}^2 \to \mathbb{R}^3, \qquad \rho(u, v) = (u + v, u - v, u^3).$$ [**5**]

(b) Show that the following is a surface:

$$Q = \{(x, y, z) \in \mathbb{R}^3 \mid x - y = 1\}.$$ [**6**]

Briefly explain your reasoning.

(c) Find a parametrisation of the surface,

$$Z = \{(x, y, z) \in \mathbb{R}^3 \mid z = e^x + e^{2y}\},$$

such that its image is all of Z. **Be sure to specify its domain.** [**5**]

Question 5 [17 marks].

(a) Sketch the following curve:

$$C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^4 = 1\}.$$

(Just the approximate shape of C is fine; you do not need to be very precise.) **[5]**

(b) Using the method of Lagrange multipliers, find the maximum and minimum values of the function

$$a : \mathbb{R}^2 \to \mathbb{R}, \qquad a(x, y) = x + y^2,$$

subject to the constraint

$$x^2 + y^4 = 1.$$ **[12]**

Question 6 [10 marks].

(a) Is the following an open subset of $\mathbb{R}^2$:

$$A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1,\ y \geq 0\}?$$

Briefly justify your answer. **[5]**

(b) Compute the divergence of the following vector field on $\mathbb{R}^3$:

$$\mathbf{G}(x, y, z) = (xe^x + yz,\ y^4,\ \cos z + \sin z)_{(x,y,z)}.$$ **[2]**

(c) Bob wants to integrate a vector field $\mathbf{F}$ (on $\mathbb{R}^3$) over a cylinder of height 1,

$$S = \{(x, y, z) \in \mathbb{R}^2 \mid x^2 + y^2 = 1,\ 0 < z < 1\},$$

which is also assigned the **outward-facing** orientation. Applying the **divergence theorem**, Bob obtains

$$\iint_S \mathbf{F} \cdot d\mathbf{A} = \iiint_V (\nabla \cdot \mathbf{F})\, dx\, dy\, dz,$$

where V is the interior of the cylinder,

$$V = \{(x, y, z) \in \mathbb{R}^2 \mid x^2 + y^2 < 1,\ 0 < z < 1\}.$$

As a tutor for **MTH5113**, you see that Bob's answer is incorrect, and you decide that Bob must lose some marks for this. Where did Bob make a mistake? **[3]**

End of Paper.