Queen Mary
University of London

Main Examination period 2019

MTH5112: Linear Algebra 1

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Examiners: I. Tomašić, T. Popiel

Question 1. [10 marks] Consider the linear system

$$\begin{array}{rcrcrcrcl} x_1 & - & x_2 & + & x_3 & - & x_4 & = & 1 \\ -x_1 & + & 2x_2 & - & 2x_3 & + & 3x_4 & = & 2 \\ 2x_1 & & & + & x_3 & + & 5x_4 & = & 3 \end{array}$$

(a) Write down the augmented matrix of the system. **[2]**

(b) Bring the augmented matrix to **reduced** row echelon form (RREF). Indicate which elementary row operation you use at each step. **[5]**

(c) Identify the leading and the free variables, and write down the solution set of the system. **[3]**

Question 2. [15 marks]

(a) Explain what it means for a matrix M to be **invertible** and what is meant by the **inverse** of M. **[4]**

(b) Suppose M and N are invertible matrices of the same size. Is it necessarily true that $M+N$ is also invertible? Give a proof or a counterexample. **[3]**

(c) Let

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

Compute A^2, A^3, A^{2019} and A^{-1}. **[8]**

Question 3. [15 marks] Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 3 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 5 \\ 2 & 3 & 4 & 5 \end{pmatrix}.$$

(a) Calculate $\det(A)$. **Hint:** consider performing some elementary row operations. **[4]**

(b) Is A an invertible matrix? Justify your answer. **[2]**

(c) Denote by $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ the columns of A, considered as vectors in $\mathbb{R}^4$.

(i) Are vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ linearly independent? Justify your answer. **[3]**

(ii) Do vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ span $\mathbb{R}^4$? Justify your answer. **[3]**

(iii) Do vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ span $\mathbb{R}^4$? Justify your answer. **[3]**

Question 4. [20 marks]

(a) Give the definition of a **subspace** of a vector space. **[4]**

(b) Give the definition of a **basis** for a vector space. **[2]**

(c) Let

$$H = \left\{ A \in \mathbb{R}^{2\times 2} \, : A^T + A = O \right\}.$$

(i) Show that H is a subspace of $\mathbb{R}^{2\times 2}$. **[4]**

(ii) Find a basis for H and determine $\dim(H)$. **[4]**

(d) Let $B \in \mathbb{R}^{m\times n}$.

(i) Define the **nullspace** $N(B)$. **[2]**

(ii) Prove that $N(B)$ is a subspace of $\mathbb{R}^n$. **[4]**

Question 5. [12 marks]

(a) State the Rank-Nullity Theorem. **[2]**

(b) Let

$$A = \begin{pmatrix} 1 & -1 & 3 & 1 & 2 \\ 4 & -4 & 12 & 6 & 0 \\ -3 & 3 & -9 & -4 & -2 \end{pmatrix}.$$

(i) Find bases for $\mathrm{row}(A)$, $\mathrm{col}(A)$ and $N(A)$. **[7]**

(ii) Determine the rank and nullity of A, and verify that the Rank-Nullity Theorem holds for the above matrix A. **[3]**

Question 6. [18 marks] Let

$$A = \begin{pmatrix} 2 & 3 & 3 \\ 0 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix}.$$

(a) Show that $\mathbf{v}_1 = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector of A and find the corresponding eigenvalue. **[4]**

(b) Find the characteristic polynomial of A and factorise it. **Hint:** the answer to (a) may be useful. **[4]**

(c) Determine all eigenvalues of A and find bases for the corresponding eigenspaces. **[6]**

(d) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. **[4]**

Question 7. [**10 marks**] Consider the least squares problem $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}.$$

(a) Write down the corresponding normal equations. [**4**]

(b) Determine the set of least squares solutions to the problem. [**3**]

(c) Let $H = \mathrm{col}(A)$ be the column space of A. Find the best approximation of $\mathbf{b}$ in H. [**3**]

End of Paper.