Queen Mary
University of London

MTH5112: Linear Algebra I

Duration: 2 hours

Date and time: 13 May 2016, 10:00

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You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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Examiner(s): C. Beck

Question 1.

(a) Consider the following matrix:

$$\begin{pmatrix} 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Is this matrix

i) in row echelon form?

ii) in reduced row echelon form?

iii) symmetric?

iv) an element of $\mathbb{R}^{5\times 4}$?

Answer with yes or no.

[**4**]

(b) Consider the linear system

$$\begin{array}{rcrcrcrcr} x_1 & - & 2x_2 & - & x_3 & + & x_4 & = & 1 \\ -x_1 & + & x_2 & & & + & x_4 & = & 2 \\ 2x_1 & - & 2x_2 & & & - & x_4 & = & -4 \end{array}$$

(i) Write down the augmented matrix of the system.

(ii) Bring the augmented matrix to row echelon form. Indicate which elementary row operation you use at each step.

(iii) Identify the leading and the free variables, and write down the solution set of the system.

[**8**]

Question 2.

(a) Let

$$A = \begin{pmatrix} 0 & -1 & 2 \\ -3 & 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 \\ -2 & 0 \end{pmatrix}.$$

For each of the products A^2, AB, BA, B^2, state whether or not it exists; if it exists then evaluate it. [**4**]

(b) Explain what it means for a matrix M to be *invertible* and what is meant by the *inverse* of M. [**4**]

(c) Show that if M and N are invertible matrices of the same size then MN is invertible and

$$(MN)^{-1} = N^{-1}M^{-1}.$$

[**4**]

Question 3. Consider the least squares problem $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}.$$

Write down the corresponding normal equations and determine the set of least squares solutions. [**7**]

Question 4. Let

$$A = \begin{pmatrix} 2 & 0 & 5 & 0 \\ 1 & 0 & 3 & 0 \\ -7 & 2 & 9 & 6 \\ 8 & 0 & 4 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ -5 \\ 9 \end{pmatrix}.$$

(a) Calculate $\det(A)$. [**3**]

(b) Using (a) deduce that the system $A\mathbf{x} = \mathbf{b}$ where $\mathbf{x} = (x_1, x_2, x_3, x_4)^T$ is consistent and determine x_3 using Cramer's rule. [**4**]

(c) Let B and C belong to $\mathbb{R}^{7\times 7}$. Suppose that $\det(B^2C) = -27$ and that B is obtained from C be adding 3 times column 2 to column 1. Find $\det(B)$ and $\det(C)$. [**5**]

Question 5. Let $H = \left\{ A \in \mathbb{R}^{2\times 2} \,\middle|\, A^T = A \right\}$.

(a) Explain what is meant by a *subspace* of a vector space. [**4**]

(b) Show that H is a subspace of $\mathbb{R}^{2\times 2}$. [**4**]

(c) Explain what is meant by a *basis* for a vector space. [**4**]

(d) Find a basis of H and determine $\dim H$. [**4**]

Question 6.

(a) Given a matrix $A \in \mathbb{R}^{m\times n}$, briefly explain what is

i) $\mathrm{row}(A)$

ii) $\mathrm{col}(A)$

iii) the nullspace $N(A)$

iv) $\mathrm{rank}\, A$

v) $\mathrm{nul}\, A$ [**5**]

(b) State the Rank-Nullity Theorem. [**2**]

(c) Let

$$A = \begin{pmatrix} 1 & -1 & 3 & 1 & 2 \\ 4 & -4 & 12 & 6 & 0 \\ -3 & 3 & -9 & -4 & -2 \end{pmatrix}.$$

By bringing the matrix A into row echelon form, find bases for $\mathrm{row}(A)$, $\mathrm{col}(A)$ and $N(A)$. Determine the rank and nullity of A, and verify that the Rank-Nullity Theorem holds for the above matrix A. [**6**]

Question 7. Consider the following vectors in $\mathbb{R}^4$

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 0 \\ 1 \\ -3 \\ 0 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 1 \\ -7 \\ -2 \\ 5 \end{pmatrix},$$

and let $H = \mathrm{Span}\,(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$.

(a) Show that the vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are linearly independent. [**4**]

(b) Use the Gram Schmidt process to determine an orthogonal basis of H. [**4**]

(c) Using (b) determine the vector in H that is closest to $\mathbf{y}$. [**4**]

Question 8. Let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & -2 & 1 \end{pmatrix}.$$

(a) Explain what is meant by an *eigenvalue* and an *eigenvector* of a matrix. [**4**]

(b) Find the characteristic polynomial of A and factorise it. [**4**]

(c) Determine all eigenvalues of A and find bases for the corresponding eigenspaces. [**4**]

(d) Is A diagonalisable? Give reasons for your answer. [**4**]

End of Paper.