Queen Mary University of London

Main Examination period 2023 – May/June – Semester B

MTH6155: Financial Mathematics II

Duration: 2 hours

The exam is intended to be completed within **2 hours**. However, you will have a period of **4 hours** to complete the exam and submit your solutions.

You should attempt ALL questions. Marks available are shown next to the questions.

All work should be **handwritten** and should **include your student number**. Only one attempt is allowed – **once you have submitted your work, it is final**.

In completing this assessment:

- You may use books and notes.
- You may use calculators and computers, but you must show your working for any calculations you do.
- You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.
- You must not seek or obtain help from anyone else.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;

Examiners: I. Goldsheid, E. Katirtzoglou

1. The following convention is used in this paper. If $Y(t)$ is a random process then Y_t may be used to describe the same process; a similar convention applies to any other random process. In particular, throughout this paper both $W(t)$ and W_t denote the standard Wiener process.
2. $\tilde{\mathbb{E}}$ denotes the expectation over a risk-neutral probability.
3. Time involved in calculations should be expressed in years. E. g., 3 months should be converted into 0.25 years.
4. The precision of calculations should be to 3 decimal places.
5. You may use without proof the following equalities. If $X \sim \mathcal{N}(0, \sigma^2)$ then $\mathbb{E}(e^X) = e^{\frac{\sigma^2}{2}}$. In particular, $\mathbb{E}(e^{bW_t}) = e^{\frac{b^2}{2}t}$, where b is any real number.

Question 1 [18 marks]. This question is about the Wiener process and the geometric Brownian motion.
Remark. You are reminded that two random variables X_1, X_2 with joint normal distribution are independent if and only if $\mathrm{Cov}(X_1, X_2) = 0$.

(a) The random process $Y(t),\ t \geqslant 0$, is defined by

$$Y(t) = tW(t^{-1}),$$

where $W(t),\ t \geqslant 0$, is the standard Wiener process.

(i) Prove that $\mathrm{Cov}(Y_t, Y_s) = \min(t, s)$, where $t > 0,\ s > 0$. **[5]**

(ii) For time moments $0 < s < t < u < v$, compute the covariance of the increments $Y_t - Y_s$ and $Y_v - Y_u$. Are the increments of the process $Y(t),\ t > 0$, independent? **[5]**

(b) Consider the geometric Brownian motion of the form $S(t) = S_0 e^{\mu t + \sigma W(t)}$. For $t \geqslant 0$, compute the expectation of the product $S(t/3)S(t)$. **[8]**

Question 2 [12 marks]. This question is about the Arbitrage Theorem.

(a) State the Arbitrage Theorem. **[3]**

(b) Three players A, B, and C are competing in a game.

If you bet £1 on A and A wins, then you are paid £3 (your pure gain is £2). If A doesn't win then you loose your £1.

If you bet £1 on B, and B wins you are paid £4. If B doesn't win then you loose your £1.

If you bet £1 on C, and C wins then you are paid £x. If C doesn't win then you loose your £1.

(i) Write down the return functions for the three possible bets. **[3]**

(ii) For what value of x there would be no arbitrage in this game? **[6]**

Question 3 [19 marks]. The price of a share is described by a geometric Brownian motion S_t, $0 \leqslant t \leqslant T$, with parameters $s = S(0),$, μ, and σ. The continuously compounded interest rate is r.

(a) A derivative on this share has expiration time T and a payoff function $R(T) = S_T^{\frac{1}{3}}$. Find the formula for the price C of this derivative. **[5]**

(b) The trader of the derivative described in (a) has to devise a hedging strategy so that to meet his financial obligation at time T. The hedging portfolio should consist of underlying shares and of money deposited in the bank.

(i) In the case of the derivative described above, derive the formulae allowing one to compute the total capital of the hedging portfolio, the number of shares in the portfolio, and the amount of money deposited in the bank at time t, $0 \leqslant t \leqslant 1.5$. **[6]**

(ii) Suppose now that $s = S(0) = £28$, $\mu = 0.2$, $\sigma = 0.25$, the expiration time is 15 months, and the continuously compounded interest rate is $r = 5\%$. Suppose that after 9 months the price of the share fell down to £25.

What should be the total value of the hedging portfolio in 9 months from now?

How many shares should be in the portfolio and how much money should be deposited in the bank? **[8]**

Question 4 [22 marks].

(a) Y_t is a random process defined for $t \geqslant 0$ by

$$Y_t = \int_0^t s^{3/2} dW_s.$$

(i) Compute the variance of this process and state the distribution of Y_t. **[4]**

(ii) Compute the expectation $\mathbb{E}\left(e^{2Y(t)}\right)$. **[5]**

(b) A random variable V is defined by

$$V = \int_1^2 s^{\frac{3}{2}} W_s dW_s.$$

Compute the expectation and the variance of V. **[4]**

(c) Compute the stochastic integral $\int_1^3 W_s^3 \, dW_s$ in terms of a function of W_t and the ordinary integral of a function of W_s. **[4]**

(d) The random process X_t, $t \geqslant 0$, satisfies the following stochastic differential equation:

$$dX_t = X_t^2 dt + tX_t dW_t.$$

A new process Y_t is defined by $Y_t = \ln(X_t^2 + t^2)$. Compute the stochastic differential dY_t. **[5]**

Question 5 [17 marks]. The following stochastic differential equation describes the behaviour of the price S_t, $t \geqslant 0$ of a share.

$$dS_t = (a + b\sin(t))S_t dt + \sigma S_t dW_t, \quad (1)$$

where a, b, σ are constants such that $a > 0$, $|b| < a$, and $\sigma > 0$.

(a) Compute the differential of the function $F(t)$ defined by $F(t) = \ln(S_t)$. **[5]**

(b) Solve the equation for S_t with the initial value $S(0) = S_0$ **[6]**

(c) Compute $\mathbb{E}(\ln(S_t))$ and $\mathrm{Var}(\ln(S_t))$ and state the distribution of $\ln(S_t)$. **[6]**

Question 6 [12 marks]. This question is about the Merton model.
The total capital $F(t)$ of a company follows the geometric Brownian motion with parameters $\mu = 0.2$ and $\sigma = 0.3$. At present, the total capital of the company is £4 million.
The company has just sold zero-coupon bonds with the total nominal value of £2.5 million which it is supposed to repay in 15 months' time.
The continuously compounded annual interest rate $r = 5\%$.
Within the framework of the **Merton model**, establish the following.

(a) How much money did the company raise by selling the bonds? [**7**]

(b) What is the probability that the company would not default on its promise to bond holders? [**5**]

End of Paper – An appendix of 1 page follows.

Table of the cumulative standard normal distribution

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt, \quad \Phi(-x) = 1 - \Phi(x)$$

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

End of Appendix.