Queen Mary University of London

Examination period 2018

MTH6155, MTH6155P: Financial Mathematics II

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Examiners: I. Goldsheid, C. Beck

The following convention is used in this paper. If $Y(t)$ is a random process then Y_t may be used to describe the same process; a similar convention applies to any other random process. In particular, both $W(t)$ and W_t denote the standard Wiener process.

Question 1. [**15 marks**] Suppose that Y_t is a Brownian motion.

(a) State the definition of a **Wiener process**. [**6**]

(b) State the definition of a **Brownian motion** with parameters μ and σ. [**3**]

(c) Prove that if $Y_t,\ t \geqslant 0$, is the Brownian motion with the drift parameter μ and volatility parameter σ then $\mathrm{Cov}(Y_t, Y_s) = \sigma^2 \min(s,t)$.

Remark. You may use without proof the equality $\mathrm{Cov}(W_t, W_s) = \min(s,t)$. [**6**]

Question 2. [**18 marks**] You are reminded that within the framework of the Black-Scholes model the price of a **European call option** with the strike price K and expiration time T is given by

$$C(S, K, \sigma, r, T) = S\Phi(\omega) - Ke^{-rT}\Phi(\omega - \sigma\sqrt{T}), \qquad (1)$$

where

$$\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-t^2/2}\,\mathrm{d}t \quad \text{and} \quad \omega = \frac{\log\frac{S}{K} + rT}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}.$$

(a) State the definitions of a European **call option** with strike price K and expiration time T and of a European **put option** with strike price K and expiration time T. [**4**]

(b) Explain the meaning of the parameters S and r in (1). [**3**]

(c) Let C be the price of a European call option and P be the price of a European put option on the same underlying share. Suppose that these options have the same strike price K and the same expiry time T. The continuously compounded interest rate is r. State the call-put parity formula. [**3**]

(d) Prove that the no-arbitrage price of a **European put option** with the strike price K and expiry time T is given by the following formula:

$$P = Ke^{-rT}\Phi(\sigma\sqrt{T} - \omega) - S\Phi(-\omega).$$

Hints: Use (1), the call-put parity formula, and the fact that $\Phi(-x) = 1 - \Phi(x)$. [**8**]

Question 3. **[14 marks]** Consider the Black-Scholes model. Let $S(t)$ be the price of a share at time $t \geqslant 0$ which is driven by a geometric Brownian motion with parameters $S,\ \mu,\ \sigma$, that is $S(t) = Se^{\mu t+\sigma W(t)}$. Let r be the continuously compounded interest rate.

(a) Suppose that a derivative has a return function $R(S(T))$ (that is, the sum of £$R(S(T))$ is paid to the owner of the derivative at time T). State the theorem which allows one to compute the no-arbitrage price of this derivative. [**4**]

(b) Compute the no-arbitrage price of a derivative with $R(S(T)) = \sqrt{S(T)}$.

Remark The fact that $\mathbb{E}\left(e^{aW(t)}\right) = e^{\frac{a^2 t}{2}}$, where a is any real number may be used without proof. [**10**]

Question 4. **[5 marks]** What is the definition of implied volatility? [**5**]

Question 5. **[19 marks]** Denote by $S(t)$ the price of a share at time $t,\ 0 \leqslant t \leqslant T$. Suppose that you have an American call option on this share with strike price K and expiration time T. The interest rate (compounded continuously) is $r > 0$. No dividends are paid.

(a) Explain the difference between a European call option and an American call option. [**3**]

(b) Explain what it means to short-sell a share. [**3**]

(c) Suppose that $S(0) > K$. Consider the following two strategies.

Strategy 1. Exercise the call option at time $t = 0$ and deposit $S(0) - K$ that you gain in a bank.

Strategy 2. Do two things: first, short-sell the share and deposit $S(0)$ in a bank; second, keep the call option until the expiration time T.

(i) Prove that the second strategy is at least as good as the first one (no matter what the price $S(T)$ is). [**10**]

(ii) Given that no dividend is paid and the interest rate $r \geqslant 0$, is it ever optimal to exercise an American call option before the expiration time? [**3**]

Question 6. [**8 marks**]

(a) Use Ito's lemma to show that $d(W_t^2) = 2W_t dW_t + dt$. [**4**]

(b) Show that $\int_0^t W_s\, dW_s = \frac{1}{2}W_t^2 - \frac{t}{2}$.
Hint Use the relation stated in (a) or any other method you may know. [**4**]

Question 7. [**21 marks**] Consider the Vasicek model according to which the interest rate $r(t)$ is governed by the stochastic differential equation

$$dr(t) = -a(r(t) - b)dt + \sigma dW_t \quad \text{with} \quad r(0) = r_0,$$

where W_t is the Wiener process, and $a > 0$, $b > 0$ are constants. Define a new function $U(t) = e^{at}\,(r(t) - b)$.

(a) Compute the differential $dU(t)$ and thus prove that

$$dU(t) = \sigma e^{at} dW_t. \tag{2}$$

Hint Apply the chain rule to obtain this result. [**8**]

(b) Use (2) to compute $U(t)$ and prove that

$$r(t) = b + (r_0 - b)e^{-at} + \sigma e^{-at} \int_0^t e^{as} dW_s.$$ [**7**]

(c) State the distribution of $r(t)$ and compute its parameters in the case when $a = 0.5$, $b = r_0 = 0.03$, $\sigma = 0.01$, and $t = 1$. [**6**]

End of Paper.