Queen Mary
University of London

Main Examination period 2018

MTHM750 / MTH750U / MTH750P: Graphs and Networks

Duration: 3 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Complete all rough work in the answer book and cross through any work that is not to be assessed.

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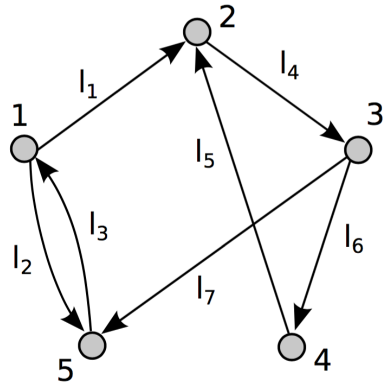
Examiners: V. Latora, G. Bianconi

Question 1. [41 marks]
Consider the directed graph G_1, with $N = 5$ nodes and $K_1 = 7$ arcs, shown below:



(a) Write down the arc list, the adjacency matrix, A_1, and the incidence matrix, B_1, of the graph. Can you say that the graph is directed by only looking at its adjacency matrix? **[9]**

(b) Is graph G_1 weakly connected? Is G_1 strongly connected? Find the matrix D_1 of distances between the nodes of the graph. **[5]**

(c) Consider now the undirected graph G_2 obtained from G_1 by connecting a pair of nodes i and j in G_2 if at least one of the two arcs, from i to j or from j to i, exists in G_1. Write down the adjacency matrix, A_2, of graph G_2, the number of links in G_2, and the matrix of node distances, D_2. Write down the degree sequences and the degree distributions of both G_1 and G_2. **[11]**

(d) State the definitions of node clustering coefficients and that of clustering coefficient of a graph. State the definition of characteristic path length of a graph. Evaluate the clustering coefficient of graph G_2 and its characteristic path length. **[8]**

(e) State the definition of node closeness centrality and that of node betweenness centrality. Evaluate the normalised closeness centrality and the normalised betweenness centrality of node 3 both in graph G_1 and in graph G_2. **[8]**

Question 2. [29 marks]
Consider an ensemble of Erdős-Renyí (ER) random graphs $G^{\mathrm{ER}}_{N,p}$, where N is the number of nodes in the graphs and p is the probability to connect each pair of nodes. Assume $N \to \infty$ and that the connection probability scales, as a function of N, as $p(N) = zN^{-1}$, where z is a non negative real constant.

(a) Determine the expression for the node degree distribution p_k of the graphs in the ensemble. Approximate the degree distribution by a Poisson distribution, and express the values of the average node degree $\langle k \rangle$, and the standard deviations, σ_k, of node degrees (averages are performed over graph nodes) as functions of z **[6]**

(b) Find an expression for the average number of cycles of length three in the graphs of the ensemble as a function of z. **[5]**

(c) Define $S = \overline{s_1}/N$, where $\overline{s_1}$ is the expected size of the largest component in the graphs of the ensemble (the average is performed over the graphs in the ensemble). Show that S satisfies the self-consistency equation:

$$S = 1 - e^{-zS}$$

[5]

(d) Solve graphically the equation above, and prove that $S \neq 0$, i.e. the system has a non-zero giant component, when the average node degree is larger than a critical value $\langle k \rangle_c = 1$. **[7]**

(e) State the Molloy-Reed criterion. Show that the criterion returns the same result as in point (d), when the degree distribution considered is Poisson. **[6]**

Question 3. [30 marks]
Consider the following non-linear modification to the Barabási-Albert model. Given three positive integers $N \gg 1$, n_0 and m (with $m \leqslant n_0$), and a real number $\beta \in [-1,1]$, grow a graph, by starting at time $t = 0$ with a complete graph with n_0 nodes, and by iteratively repeating at time $t = 1,2,3,\ldots,N-n_0$, the following two steps:

(1) A new node, labeled by the index n, being $n = n_0 + t$, is added to the graph. The node arrives together with m edges.

(2) The m edges link the new node to m different nodes already present in the system. The probability $\Pi_{n\to i}$ that a new edge links the new node n to node i (with $i = 1,2,\cdots,n-1$) is:

$$\Pi_{n\to i} = \frac{k_{i,t-1}^{\beta}}{\sum_{l=1}^{n-1} k_{l,t-1}^{\beta}}$$

where $k_{i,t}$ is the degree of node i at time t.

(a) Write down the expression for the number of nodes, n_t, the number of links, l_t, and the average node degree as functions of time t. **[6]**

(b) Find the average node degree $\langle k \rangle$ when $N \to \infty$ **[4]**

(c) From now on, fix $n_0 = 3$ and $m = 3$. Write down the rate equations of the model, i.e. the equations for $\overline{n}_{k,t}$, where $\overline{n}_{k,t}$ denotes the average number of nodes with degree k present in the graph at time t. Averages are performed over infinite realizations of the growth process with the same parameters N, n_0, m and β. **[6]**

(d) In the case $\beta = 1$, determine the stationary degree distribution p_k (i.e. the limit of $p_{k,t} = \overline{n}_{k,t}/n_t$ when $t \to \infty$), and show that it is scale free with an exponent γ equal to 3. **[9]**

(e) In the case $\beta = 1$, write down and solve the differential equation governing the time evolution of the average degree $\overline{k_i}(t)$ of a node i introduced at time $t_i = 100$, in the so-called mean-field approximation. **[5]**

End of Paper.