Queen Mary
University of London

Main Examination period 2017

MTHM750 / MTH750U / MTH750P: Graphs and Networks

Duration: 3 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks available are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

Possession of unauthorised material at any time when under examination conditions is an assessment offence and can lead to expulsion from QMUL. Check now to ensure you do not have any notes, mobile phones, smartwatches or unauthorised electronic devices on your person. If you do, raise your hand and give them to an invigilator immediately.

It is also an offence to have any writing of any kind on your person, including on your body. If you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms, it shall be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. A mobile phone that causes a disruption in the exam is also an assessment offence.

Exam papers must not be removed from the examination room.

Examiners: V. Latora, G. Bianconi

Question 1. [39 marks]

Consider the graph G with $N = 5$ nodes described by the adjacency matrix:

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) Draw the graph. Is the graph directed? Is it connected? How are these two properties of the graph related to the properties of the adjacency matrix? **[7]**

(b) Find the nodes with the largest in-degree and the ones with the largest out-degree. Write down the in-degree distribution p_k^{in} and the out-degree distribution p_k^{out}. **[6]**

(c) State the definition of the α-*centrality* of a node of a graph. Find the normalized α-centrality for all the nodes of graph G. **[11]**

(d) Derive a general expression for the number of directed triangles n_Δ in terms of the adjacency matrix of a directed graph, and calculate the number of directed triangles in G. **[7]**

(e) Find the matrix of distances between nodes of the graph G. Find the matrix of distances between the nodes of the graph G' obtained from graph G by considering all the arcs of G as undirected. **[8]**

Question 2. [28 marks]
Consider the configuration model and construct an ensemble of random graphs with $N = 10000$ nodes, power-law degree distribution $p(k) = ck^{-\gamma}$, with $c > 0$ and $\gamma = 3$, and where the smallest and largest degree in each graph is respectively equal to $k_{\min}$ and $k_{\max}$. In the following, treat the degree k as a real positive number, i.e. work in the so-called continuous-k approximation.

(a) Determine the value of the normalisation constant c. [**4**]

(b) Express the first and second order moments of the degree distribution, $\langle k \rangle$ and $\langle k^2 \rangle$, as functions of $k_{\min}$ and $k_{\max}$. [**9**]

(c) Write down the expression for the probability $q(k)$ to find a node of degree k by following a link of the graph (i.e the probability to arrive at a node of degree k by selecting a link at random with uniform probability, and then considering one of the two end nodes of the link). [**4**]

(d) What are the values of the average degree of a node, and of the average degree of its neighbours in the case in which $k_{\min} = 1$, $k_{\max} = 1000$? [**5**]

(e) State the Molloy-Reed criterion. Do the graphs in the ensemble considered in point (d) have a giant connected component? [**6**]

Question 3. [33 marks]

Consider the following model to grow graphs.
Given three positive integers $N \gg 1$, $n_0 = 10$ and $m = 2$, and a real number a $(-m \leqslant a)$, the graph grows, starting at time $t = 0$ with a complete graph with n_0 nodes, and by iteratively repeating at time $t = 1, 2, 3, \ldots, N - n_0$, the two steps:

(1) A new node, labelled by the index n, being $n = n_0 + t$, is added to the graph. The node arrives together with m edges.

(2) The m edges link the new node to m different nodes already present in the system. The probability $\Pi_{n \to i}$ that a new edge links the new node n to node i (with $i = 1, 2, \cdots, n-1$) is:

$$\Pi_{n \to i} = \frac{k_{i,t-1} + a}{\sum_{l=1}^{n-1}(k_{l,t-1} + a)}$$

where $k_{i,t}$ is the degree of node i at time t.

(a) Find an expression for the number of nodes, n_t, and the number of links, l_t, as a function of time t. **[5]**

(b) What is the final number of nodes and links in the graph, and what is the average node degree $\langle k \rangle$, when $N \to \infty$? **[6]**

(c) Write down the rate equations of the model, i.e. the equations for $\overline{n}_{k,t}$, where $\overline{n}_{k,t}$ denotes the average number of nodes with degree k ($k \geqslant m$) present in the graph at time t. The average, as usual, is performed over infinite realisations of the growth process with the same parameters **[7]**

(d) Solve the rate equations in the case $a = 0$, and find the corresponding stationary degree distribution p_k. Notice that p_k is the limit of $p_{k,t} = \overline{n}_{k,t}/n_t$ when $t \to \infty$. **[10]**

(e) Does the model produce scale-free networks in the case $a = 0$? If so, what is the value of the degree exponent γ? **[5]**

End of Paper.