Queen Mary
University of London

Main Examination period 2023 – May/June – Semester B

MTH745: Further Topics in Algebra

Duration: 3 hours

The exam is intended to be completed within **3 hours**. However, you will have a period of **4 hours** to complete the exam and submit your solutions.

You should attempt ALL questions. Marks available are shown next to the questions.

All work should be **handwritten** and should **include your student number**. Only one attempt is allowed – **once you have submitted your work, it is final**.

In completing this assessment:

- You may use books and notes.
- You may use calculators and computers, but you must show your working for any calculations you do.
- You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.
- You must not seek or obtain help from anyone else.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;

Examiners: Navid Nabijou

Important: All your answers must be justified. Unless the question explicitly indicates otherwise, you may use any result from lectures, but you must state it clearly. All rings are commutative and have an identity element.

Question 1 [25 marks].

(a) Give an example of an integral domain which is not a field. [**5**]

(b) For each of the following statements, give either a proof or a counterexample.

(For the counterexamples, you do **not** need to justify that the rings you provide are integral domains, PIDs, fields, etc.)

(i) A subring of an integral domain is an integral domain. [**3**]

(ii) A subring of a field is a field. [**3**]

(iii) A subring of a PID is a PID. [**3**]

(c) Let R be a ring and consider for every natural number $n \geq 1$ the set

$$\mu_n(R) := \{a \in R \mid a^n = 1\}.$$

(i) Prove that $\mu_n(R)$ is an abelian group under multiplication. [**3**]

(ii) Find a ring R such that $\mu_n(R) \cong C_n$ for all $n \geq 1$. Here C_n denotes the cyclic group of order n. [**4**]

(iii) Find a ring R such that $\mu_n(R) = \{1\}$ for all $n \geq 1$. [**4**]

Question 2 [25 marks].

(a) (i) Consider the following polynomials in $\mathbb{Q}[x]$:

$$f(x) = x^4 + 7x^3 + 5, \quad g(x) = x^2 + 2.$$

Use the division algorithm to write $f(x) = q(x)g(x) + r(x)$ with $\deg r < \deg g$. [**5**]

(ii) Consider the following polynomials in $\mathbb{Q}[x]$:

$$f_1(x) = x^3 + x^2 + x + 1, \quad f_2(x) = x^3 - x^2 + x - 1.$$

Find a polynomial $h \in \mathbb{Q}[x]$ such that there is an equality of ideals:

$$(f_1, f_2) = (h).$$ [**5**]

(iii) Prove that the polynomial

$$f(x) = x^3 + 2x^2 + 1 \in \mathbb{F}_3[x]$$

is irreducible. [**5**]

(b) Consider the polynomial

$$f(x) = x^4 - 2 \in \mathbb{Q}[x].$$

Let $F \subseteq \mathbb{C}$ be the splitting field for f over $\mathbb{Q}$. Find $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$ such that

$$F = \mathbb{Q}(\alpha_1, \ldots, \alpha_k)$$

and calculate $[F\colon \mathbb{Q}]$ carefully. [**10**]

Question 3 [25 marks].

(a) Carefully calculate the degrees of the following field extensions:

(i) $\mathbb{Q}(\sqrt[4]{5})|\mathbb{Q}$. [**5**]

(ii) $\mathbb{Q}(\sqrt{\sqrt{6}+4})|\mathbb{Q}$. [**5**]

(iii) $\mathbb{Q}(\sqrt{7}+i, i)|\mathbb{Q}$. [**5**]

(b) Which of the following field extensions are normal?

(i) $\mathbb{Q}(\sqrt[3]{2})|\mathbb{Q}$. [**3**]

(ii) $\mathbb{Q}(\zeta_3)|\mathbb{Q}$ where $\zeta_3 = \mathrm{e}^{2\pi i/3} \in \mathbb{C}$. [**3**]

(iii) $\mathbb{Q}(\sqrt[4]{5})|\mathbb{Q}(\sqrt{5})$. [**4**]

Question 4 [25 marks]. Consider the field

$$F = \mathbb{Q}(\sqrt{2}, \sqrt[3]{2}, \zeta_3)$$

where $\zeta_3 = e^{2\pi i/3} \in \mathbb{C}$. Let $G = \mathrm{Aut}(F|\mathbb{Q})$.

(a) Prove that $F|\mathbb{Q}$ is normal. [**8**]

(b) Calculate $|G|$. You may assume without proof that $\sqrt{2} \notin \mathbb{Q}(\sqrt[3]{2}, \zeta_3)$. [**10**]

(c) Use the Fundamental Theorem of Galois Theory to find a normal subgroup H of G with $H \cong C_2$, the cyclic group of order 2. [**4**]

(d) Use the Fundamental Theorem of Galois Theory to prove that G/H is not abelian. [**3**]

End of Paper.