Queen Mary University of London

MTH744U / MTH744P: Dynamical Systems

Duration: 3 hours

Date and time: 18th May 2016, 14:30–17:30

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

Possession of unauthorised material at any time when under examination conditions is an assessment offence and can lead to expulsion from QMUL. Check now to ensure you do not have any notes, mobile phones, smartwatches or unauthorised electronic devices on your person. If you do, raise your hand and give them to an invigilator immediately. It is also an offence to have any writing of any kind on your person, including on your body. If you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms it shall be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. A mobile phone that causes a disruption in the exam is also an assessment offence.

Exam papers must not be removed from the examination room.

Examiner(s): W. Just

This is an OPEN BOOK exam

Question 1.

(a) Sketch the phase portrait of a flow on the circle, which has exactly three fixed points, one being linearly stable, one being linearly unstable, and one being marginal. Describe the basin of the stable fixed point. [**10**]

(b) Sketch a differentiable function f such that the differential equation $\dot{\theta} = f(\theta)$ generates a flow satisfying the conditions of part a), and write down an explicit formula for such a function f. [**6**]

(c) Does your differential equation $\dot{\theta} = f(\theta)$ have a potential $V(\theta)$? Compute such a potential, or state a reason why a potential does not exist. [**6**]

(d) Consider the differential equation $\dot{\theta} = f(\theta)^2$. What is the number of fixed points of this dynamical system? What are the linear stability properties of each fixed point? Describe the phase portrait. [**8**]

Question 2. Consider the system of differential equations

$$\dot{x} = x(1 - x^2 - y^2) - \sigma y, \quad \dot{y} = y(1 - x^2 - y^2) + \sigma x - h$$

where $h \geq 0$ and $\sigma \geq 0$ denote the parameters of the system.

(a) For the case where $h = 0$ and $\sigma > 0$, show that introducing polar coordinates (r, ϕ), where $x = r\cos\phi$ and $y = r\sin\phi$, transforms the system to the form

$$\dot{r} = r(1 - r^2) \quad , \quad \dot{\phi} = \sigma \, .$$

[**6**]

(b) Using the above polar form, or otherwise, show that the system has one fixed point and one limit cycle, and determine the stability of these. [**6**]

(c) Consider the general case $h \geq 0$ and $\sigma \geq 0$. Compute the parameter values for which the equations of motion show saddle-node bifurcations. [**12**]

(d) For $h \geq 0$ and $\sigma \geq 0$ compute the parameter values for which the equations of motion show Hopf bifurcations. Sketch the bifurcation lines of the saddle-node and of the Hopf bifurcations in a diagram. [**12**]

Question 3. Consider the system of differential equations

$$\dot{x} = x(1 - 2x^2 - y^2) - y(1 + x), \quad \dot{y} = y(1 - 2x^2 - y^2) + 2x(1 + x)\,.$$

(a) Compute the fixed points of the system of differential equations. For each fixed point determine the stability using linear stability analysis. **[8]**

(b) Consider the quantity $L = (1 - 2x^2 - y^2)^2$. Show that $dL/dt \leq 0$. **[6]**

(c) Using the results of part b), or otherwise, show that the system of equations has a limit cycle. Is the limit cycle stable or unstable? Give a reason for your answer. **[12]**

(d) Using the results of part a) and c), or otherwise, sketch the phase portrait of the system of differential equations. **[8]**

End of Paper.