M. Sci. Examination by course unit 2015

MTH 744U Dynamical Systems

Duration: 3 hours

Date and time: 27 May 2015, 10:00h–13:00h

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt all questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

This is an OPEN BOOK exam

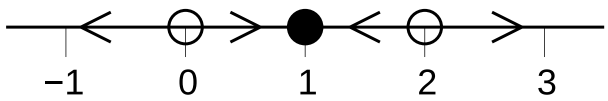
permitted:	**any printed material, e.g. books** **any handwritten notes** **photocopies of any kind**
prohibited:	**using electronic devices, e.g. calculators or mobile phones** **sharing material with other students**

Exam papers must not be removed from the examination room.

Examiner(s): W. Just

Question 1 (One-dimensional systems) **[30 marks]**

Consider a one-dimensional dynamical system which is governed by the following phase portrait:



a) Sketch an x-t diagram which is consistent with such a phase portrait. Your diagram should display solutions with initial conditions $x(0)=-1/2,0,1/2$, 1, 3/2, 2, 5/2, and 3. [8 marks]

b) State an analytic expression for a differential equation $\dot{x} = f(x)$ which is consistent with the phase portrait. [6 marks]

c) Compute the linear stability of every fixed point of the differential equation you have stated in part b). [6 marks]

d) Write down a potential $V(x)$ for the differential equation you have stated in part b). [6 marks]

e) For every stable fixed point of the differential equation stated in part b) write down the basin of attraction. [4 marks]

Question 2 (Bifurcations) **[34 marks]**

Consider the differential equation

$$\dot{x} = rx + x^3 - x^5$$

with odd right hand side, which depends on a real parameter r.

a) Using linear stability analysis, compute the range of parameter values such that the trivial fixed point, $x_* = 0$, is linearly stable, and the range of parameter values for which it is linearly unstable. If you find bifurcations of the trivial fixed point state with a reason the type of the bifurcation. [8 marks]

b) Compute the parameter values $r = r_*$ and the points $x = x_*$ in phase space where nontrivial fixed points, $x_* \neq 0$, undergo a saddle node bifurcation. [8 marks]

c) Using the results from parts a) and b), or otherwise, sketch the bifurcation diagram of the differential equation. Your diagram should indicate the stability of each fixed point. The diagram should also contain phase portraits for parameter values where phase portraits qualitatively differ. [8 marks]

d) The bifurcation values r_* computed in parts a) and b) split the parameter axis into intervals (see also the diagram constructed in part c)). State for each of these parameter intervals the number of stable and unstable fixed points. For the stable fixed points state the basins of attraction. [10 marks]

Question 3 (Two dimensional systems) **[36 marks]**

Consider the system of two differential equations:

$$\dot{x} = y - x, \qquad \dot{y} = -x + x^2$$

a) Compute the fixed points of the two dimensional system. For each fixed point perform a linear stability analysis and classify the type of fixed point. Sketch the flow in the phase plane in a small neighbourhood of each fixed point. [10 marks]

b) Show that the system given by the two differential equations is not a gradient system. [4 marks]

c) Show that the system given by the two differential equations is not a Hamiltonian system. [4 marks]

d) Using the Bendixson criterion, or otherwise, show that the system of differential equations does not have a limit cycle. [4 marks]

e) Compute the nullclines of the system of differential equations. Sketch the nullclines in the phase plane.

The nullclines partition the phase plane into different regions. For each region, and on each nullcline, indicate the direction of the flow. [8 marks]

f) Using the results from parts a), d), and e), or otherwise, sketch the full phase portrait of the two dimensional system. The phase portrait should be consistent with the diagram produced in part e). If the system has a stable fixed point then shade its basin of attraction. [6 marks]

End of Paper